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On *t*-pure and almost pure exact sequences of LCA groups

Peter Loth

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Abstract. A proper short exact sequence

$$0 \to A \xrightarrow{\phi} B \to C \to 0 \tag{$*$}$$

in the category of locally compact abelian groups is said to be *t*-pure if $\phi(A)$ is a topologically pure subgroup of B, that is, if

$$\phi(A) \cap \overline{nB} = \overline{n\phi(A)}$$

for all positive integers n. We establish conditions under which *t*-pure exact sequences split and determine those locally compact abelian groups $K \oplus D$ (where K is compactly generated and D is discrete) which are *t*-pure injective or *t*-pure projective. Calling the extension $(*)$ *almost pure* if

$$\phi(A) \cap nB \subseteq \overline{n\phi(A)}$$

for all positive integers n, we obtain a complete description of the almost pure injectives and almost pure projectives in the category of locally compact abelian groups.

1 Introduction

All groups considered in this paper are Hausdorff topological abelian groups and they will be written additively. For a group G and a positive integer n, let $nG = \{ng : g \in G\}$. If H is a subgroup of G, let $n^{-1}H = \{g \in G : ng \in H\}$. Let $\mathfrak{L}$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called *proper* if it is open onto its image, and an exact sequence

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} G_n$$

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in $\mathfrak{L}$ is called *proper exact* if each morphism ϕ_i is proper. Following Fulp and Griffith [4], we call a proper short exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ an *extension of A by C* and let $\mathrm{Ext}(C, A)$ denote the group of extensions of A by C.

Recall that a proper exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ is said to be *topologically pure* if the induced sequence

$$0 \to \overline{nA} \to \overline{nB} \to \overline{nC} \to 0$$

is proper exact for all positive integers n (cf. [7]). Evidently a topologically pure exact sequence is t-pure, but a t-pure exact sequence need not be topologically pure (see [7, Example 3.5]). A t-pure exact sequence need not be pure and a pure exact sequence need not be t-pure (see [7] or [11]).

Let $\mathrm{Tpext}(C, A)$ denote the set of elements in $\mathrm{Ext}(C, A)$ represented by t-pure exact sequences. Then the first Ulm subgroup of $\mathrm{Ext}(C, A)$ is contained in but not necessarily equal to $\mathrm{Tpext}(C, A)$ (see Theorem 2.6). If G is t-pure injective in $\mathfrak{L}$, then G has the form $R \oplus T \oplus G'$ where R is a vector group, T is a toral group and G' is a topological torsion group, but the converse fails in general as non-trivial finite groups are not t-pure injective in $\mathfrak{L}$ (cf. Theorem 2.7). Let $\mathfrak{C}$ denote the class of all LCA groups of the form $K \oplus D$ where K is a compactly generated group and D is a discrete group. Then a group $G \in \mathfrak{C}$ has the property that $\mathrm{Tpext}(X, G) = 0$ for all groups $X \in \mathfrak{L}$ if and only if G is topologically isomorphic to $I \oplus C$ where I is an injective group in $\mathfrak{L}$ and C is a topological direct product of finite cyclic groups. On the other hand, $G \in \mathfrak{C}$ satisfies $\mathrm{Tpext}(G, X) = 0$ for all groups $X \in \mathfrak{L}$ exactly if it is projective in $\mathfrak{L}$ (see Theorem 2.8). It follows that a group in $\mathfrak{C}$ is t-pure injective (t-pure projective) in $\mathfrak{L}$ exactly if it is injective (projective) in $\mathfrak{L}$ (Corollary 2.9). An LCA group is almost pure injective in $\mathfrak{L}$ exactly if it is injective in $\mathfrak{L}$ (see Corollary 3.5). Since the dual of an almost pure exact sequence is again almost pure, this implies that an LCA group is almost pure projective in $\mathfrak{L}$ if and only if it is projective in $\mathfrak{L}$ (see Corollary 3.6).

The additive groups of integers and rationals are denoted by $\mathbb{Z}$ and $\mathbb{Q}$ respectively and $\mathbb{Z}(n)$ is the cyclic group of order n. By $\mathbb{R}$ we mean the additive topological group of real numbers and we set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The Pontrjagin dual of a group $G \in \mathfrak{L}$ is

$$\hat{G} = \mathrm{Hom}(G, \mathbb{T})$$

and the annihilator of $S \subseteq G$ is written $(\hat{G}, S)$. Throughout this paper we use the term 'isomorphic' for 'topologically isomorphic', 'direct summand' for 'topological direct summand' and 'direct product' for 'topological direct product'. For details and fundamental results on locally compact abelian groups and Pontrjagin duality, we refer to Hewitt and Ross [5].

2 Topologically pure subgroups

The following result will be needed.

Theorem 2.1. *Let G be a group in $\mathfrak{L}$.*

(i) *If G is a direct product of finite cyclic groups, then* $\mathrm{Tpext}(X, G) = 0$ *for all* $X \in \mathfrak{L}$.

(ii) *If G is an infinite discrete bounded group, then there is a group $X \in \mathfrak{L}$ such that* $\mathrm{Tpext}(X, G) \neq 0$.

Proof. The statement follows from [7, Theorem 3.1 and Example 3.5].

Lemma 2.2. *A pullback of a t-pure exact sequence is t-pure. A pushout of a t-pure exact sequence need not be t-pure.*

Proof. Let $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ represent an element of $\mathrm{Tpext}(C, A)$ and assume that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & X & \xrightarrow{\nu} & Y & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

is a standard pullback diagram in $\mathfrak{L}$ (see [4]). Then

$$X = \{(y, b) \in Y \oplus B : f(y) = \psi(b)\}$$

and

$$\mu : a \mapsto (0, \phi(a)), \quad \nu : (y, b) \mapsto y, \quad \theta : (y, b) \mapsto b.$$

If n is a positive integer and $(0, b) \in \mu(A) \cap \overline{nX}$, then $b \in \phi(A) \cap \overline{nB} = \overline{n\phi(A)}$ since $\phi(A)$ is topologically pure in B and it follows that $(0, b) \in \overline{n\mu(A)}$. Therefore the top row of the diagram is t-pure and the first statement follows.

To prove that a pushout of a t-pure exact sequence need not be t-pure, let p be a prime, n a positive integer and $\{C_i : i \in \mathbb{N}\}$ a collection of cyclic groups such that every group $C_i = \langle y_i \rangle$ has order p^{n+1}. We set $y = (y_i)_{i \in \mathbb{N}}$ and $L = B \times pC$, where C is the compact group $\prod_{i \in \mathbb{N}} C_i$ and B is the subgroup $\bigoplus_{i \in \mathbb{N}} C_i + \langle y \rangle$ of C, taken discrete. It is clear that $K = \{(pb, pb) : b \in B\}$ is a discrete, hence closed, subgroup of L. To show that the sequence $E : 0 \to K \to L \to L/K \to 0$ is t-pure, let r be a positive integer and $x \in p^r L \cap K$. Then $x = p^r(b, b)$ for some $b \in B$ and there are elements $c_i \in C_i$ $(i \in \mathbb{N})$, $m \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$ such that

$$p^r b = p^r(pc_1, \ldots, pc_m, \lambda y_{m+1}, \lambda y_{m+2}, \ldots) = p^r(pc_1, \ldots, pc_m, pc_{m+1}, pc_{m+2}, \ldots).$$

Letting λ' be an integer such that $c_{m+1} = \lambda' y_{m+1}$, we set

$$z = (pc_1, \ldots, pc_m, p\lambda' y_{m+1}, p\lambda' y_{m+2}, \ldots)$$

and obtain $x = p^r(z, z) \in p^r K$. Since $p^r L = \overline{p^r L}$, it follows that the sequence E is t-pure. Now let $H = \{(pa, pa) : a \in A\}$ where $A = \bigoplus_{i \in \mathbb{N}} C_i \subset B$. Then the natural map $\alpha : K \to K/H$ induces a pushout diagram

$$\begin{array}{ccccccccc}
E\colon & 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & L/K & \longrightarrow & 0 \\
 & & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow & & \big\| & & \\
\alpha E\colon & 0 & \longrightarrow & K/H & \longrightarrow & L/H & \longrightarrow & L/K & \longrightarrow & 0.
\end{array}$$

Since $(py, py) \in K \backslash (pK + H)$, we have

$$\overline{(pL + H)} \cap K = (pB \times pC) \cap K = K \nsubseteq pK + H,$$

i.e., $\overline{p(L/H)} \cap K/H \nsubseteq p(K/H)$. Therefore αE is not t-pure.

Lemma 2.3. *Let A and C be groups in $\mathfrak{L}$.*

(i) *If A and C are in $\mathfrak{C}$, then* $\mathrm{Pext}(C, A) = \mathrm{Tpext}(C, A)$.

(ii) *If* $\mathrm{Tpext}(C, A) = 0$, *then* $\mathrm{Pext}(\hat{A}, \hat{C}) = 0$.

(iii) *If A is compact or C is discrete, then* $\mathrm{Tpext}(C, A)$ *is a group and is isomorphic to* $\mathrm{Pext}(\hat{A}, \hat{C})$.

Proof. (i) Suppose that $A, C \in \mathfrak{C}$. If E represents an element of $\mathrm{Pext}(C, A)$, then the dual sequence $\hat{E}$ represents an element of $\mathrm{Tpext}(\hat{A}, \hat{C})$ (see [7, Proposition 2.1]). Since $\hat{C} \in \mathfrak{C}$, we have $\hat{E} \in \mathrm{Pext}(\hat{A}, \hat{C})$, which implies that $E \in \mathrm{Tpext}(C, A)$. On the other hand, $\mathrm{Tpext}(C, A) \subseteq \mathrm{Pext}(C, A)$ since $A \in \mathfrak{C}$, and (i) follows. Using the fact that the dual of a pure exact sequence is t-pure once more, we conclude that (ii) holds. To prove (iii), consider the natural isomorphism $\mathrm{Ext}(C, A) \xrightarrow{\sim} \mathrm{Ext}(\hat{A}, \hat{C})$ given by $E \mapsto \hat{E}$. By [7, Corollary 2.2], E is t-pure if and only if $\hat{E}$ is pure. Consequently, $\mathrm{Tpext}(C, A)$ is a group which is isomorphic to $\mathrm{Pext}(\hat{A}, \hat{C})$.

By Lemma 2.3 and [8, Theorem 2.1] we obtain the following result.

Theorem 2.4. *Let $G \in \mathfrak{L}$ and suppose that $\{H_i : i \in I\}$ is a collection in $\mathfrak{L}$. If every group H_i is discrete, then*

$$\mathrm{Tpext}\left(\bigoplus_{i \in I} H_i, G\right) \cong \prod_{i \in I} \mathrm{Tpext}(H_i, G).$$

If every group H_i is compact, then

$$\mathrm{Tpext}\left(G, \prod_{i \in I} H_i\right) \cong \prod_{i \in I} \mathrm{Tpext}(G, H_i).$$

Lemma 2.5. *Let A and C be in $\mathfrak{L}$. If m is a positive integer such that $mA = 0$ or $mC = 0$, then $m\,\mathrm{Ext}(C, A) = 0$.*

Proof. As in the discrete case (see [3, p. 223]), the statement follows from the fact that Ext is an additive functor.

Our next result involves $\mathrm{Tpext}(C,A)$ and $\mathrm{Ext}(C,A)^1$, the first Ulm subgroup of $\mathrm{Ext}(C,A)$. As is well known, the latter group coincides with $\mathrm{Pext}(C,A)$ if A and C are discrete (see [3, Theorem 53.3]). For arbitrary LCA groups A and C, $\mathrm{Ext}(C,A)^1$ is a (possibly proper) subgroup of $\mathrm{Pext}(C,A)$ (see [8, Theorem 2.4]). Recall that a topological group G is called *compactly generated* if it contains a compact subset F such that the subgroup generated by F is G, and G is said to have *no small subgroups* if there is a neighborhood of 0 which does not contain any non-trivial subgroups.

Theorem 2.6. *Let A and C be groups in $\mathfrak{L}$. Then the following hold*:

(i) $\mathrm{Tpext}(C,A) \supseteq \mathrm{Ext}(C,A)^1$.

(ii) $\mathrm{Tpext}(C,A) \neq \mathrm{Ext}(C,A)^1$ *in general*.

(iii) *Suppose that A is compact. Then* $\mathrm{Tpext}(C,A) = \mathrm{Ext}(C,A)^1 \subseteq \mathrm{Pext}(C,A)$ *but* $\mathrm{Tpext}(C,A) \neq \mathrm{Pext}(C,A)$ *generally*.

(iv) *Suppose that* (a) *A and C are compactly generated or* (b) *A and C have no small subgroups. Then* $\mathrm{Tpext}(C,A) = \mathrm{Ext}(C,A)^1$.

Proof. The proof of (i) is similar to the proof of [8, Theorem 2.4(i)]. Assertion (ii) follows from Theorem 2.1 and Lemma 2.5. To prove (iii), assume that A is compact. By the proof of Lemma 2.3(iii), the natural isomorphism $\mathrm{Ext}(C,A) \xrightarrow{\sim} \mathrm{Ext}(\hat{A},\hat{C})$ maps $\mathrm{Tpext}(C,A)$ onto $\mathrm{Pext}(\hat{A},\hat{C}) = \mathrm{Ext}(\hat{A},\hat{C})^1$. Therefore

$$\mathrm{Tpext}(C,A) = \mathrm{Ext}(C,A)^1 \subseteq \mathrm{Pext}(C,A)$$

and the inclusion may be proper since a finite pure subgroup of an LCA group need not be topologically pure (cf. [7, Example 2.4]). Finally, (iv) follows from Lemma 2.3(i) and [8, Theorem 2.4(iii)].

Let G be a group in $\mathfrak{L}$. We call G *t-pure injective in* $\mathfrak{L}$ if it has the injective property relative to the class of t-pure exact sequences, i.e., if for every t-pure exact sequence

$$0 \to A \xrightarrow{\phi} B \to C \to 0$$

and continuous homomorphism $f : A \to G$ there is a continuous homomorphism $\bar{f} : B \to G$ such that $\bar{f}\phi = f$. Similarly, G is said to be *t-pure projective in* $\mathfrak{L}$ if it has the projective property relative to the class of t-pure exact sequences. Following Robertson [10], we call G a *topological torsion group* if $(n!)x \to 0$ for all $x \in G$.

Theorem 2.7. *Consider the following conditions for a group G in $\mathfrak{L}$*:

(i) *G is t-pure injective in $\mathfrak{L}$*;

(ii) $\mathrm{Tpext}(X,G) = 0$ *for all groups X in $\mathfrak{L}$*;

(iii) $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}} \oplus G'$ *where n is a non-negative integer*, $\mathfrak{m}$ *is a cardinal and G' is a topological torsion group*.

Then we have the following assertions: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and* (iii) $\not\Rightarrow$ (ii) $\not\Rightarrow$ (i).

Proof. Obviously (i) implies (ii). To prove (ii) $\not\Rightarrow$ (i), let

$$E : 0 \to K \to L \to L/K \to 0 \quad \text{and} \quad \alpha : K \to K/H$$

be as in the proof of Lemma 2.2. Then $K/H \cong \mathbb{Z}(p^n)$, and so by Theorem 2.1(i) we have $\operatorname{Tpext}(X, K/H) = 0$ for all $X \in \mathfrak{L}$. Now assume that K/H is t-pure injective in $\mathfrak{L}$. Since E is t-pure, there is a continuous homomorphism $\psi : L \to K/H$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & L/K & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\psi} & & & & & \\ & & K/H & & & & & & \end{array}$$

is a commutative diagram. Clearly the map

$$\psi' := L/H \to K/H, \quad x + H \mapsto \psi(x)$$

is a continuous homomorphism making the diagram

$$\begin{array}{cccccccccc} \alpha E: & 0 & \longrightarrow & K/H & \longrightarrow & L/H & \longrightarrow & L/K & \longrightarrow & 0 \\ & & & \| & \swarrow{\scriptstyle\psi'} & & & & & \\ & & & K/H & & & & & & \end{array}$$

commutative. But then αE splits, contradicting the fact that αE is not t-pure. Therefore K/H is not t-pure injective and (ii) $\not\Rightarrow$ (i) follows.

Now suppose that $\operatorname{Tpext}(X, G) = 0$ for all groups $X \in \mathfrak{L}$. By [7, Theorem 4.3] we have $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}} \oplus G'$ where G' is totally disconnected. As in the proof of [8, Theorem 2.7] we conclude that G' is a topological torsion group, and hence (ii) implies (iii). Finally, Theorem 2.1(ii) shows that (iii) does not imply (ii).

We shall now determine the groups $G \in \mathfrak{C}$ satisfying $\operatorname{Tpext}(X, G) = 0$ (resp. $\operatorname{Tpext}(G, X) = 0$) for all $X \in \mathfrak{L}$.

Theorem 2.8. *Let $G \in \mathfrak{C}$. Then the following assertions hold.*

(i) $\operatorname{Tpext}(X, G) = 0$ *for all $X \in \mathfrak{L}$ if and only if $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}} \oplus C$ where n is a nonnegative integer, $\mathfrak{m}$ is a cardinal and C is a direct product of finite cyclic groups.*

(ii) $\operatorname{Tpext}(G, X) = 0$ *for all $X \in \mathfrak{L}$ if and only if $G \cong \mathbb{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbb{Z}$ where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*

Proof. (i) Suppose that $\operatorname{Tpext}(X, G) = 0$ for all $X \in \mathfrak{L}$. Then Theorem 2.7 yields that $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}} \oplus C \oplus D$, where C is a compact totally disconnected group and D is a discrete torsion group. By Lemma 2.3(iii), we have $\operatorname{Pext}(\hat{C}, \hat{X}) \cong \operatorname{Tpext}(X, C) = 0$ for all $X \in \mathfrak{L}$, and hence $\hat{C}$ is a direct sum of cyclic groups (cf. [3, Theorem 30.2]);

therefore C is a direct product of finite cyclic groups. The group D is reduced because an extension of a quasicyclic group by an LCA group need not split (see [2, Example 6.4]). Since D is torsion and cotorsion, it is bounded (see [3, Corollary 54.4]). Further, Theorem 2.1(ii) shows that D is finite. Conversely, if $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}} \oplus C$ as in the theorem, then $\mathrm{Tpext}(X, G) = 0$ for all $X \in \mathfrak{L}$ (cf. Theorem 2.1(i), [9, Theorem 3.2] and [2, (6.34(b))]).

(ii) Now suppose that $\mathrm{Tpext}(G, X) = 0$ for all $X \in \mathfrak{L}$. By Lemma 2.3(ii), we have $\mathrm{Pext}(\hat{X}, \hat{G}) = 0$ for all $X \in \mathfrak{L}$, and from [8, Corollary 2.8] it follows that $\hat{G} \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$, so that $G \cong \mathbb{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbb{Z}$. The converse follows from [9, Theorem 3.3].

Corollary 2.9. *Let $G \in \mathfrak{C}$. Then the following assertions hold.*

(i) *G is t-pure injective in $\mathfrak{L}$ if and only if $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$ where n is a non-negative integer and $\mathfrak{m}$ is a cardinal.*

(ii) *G is t-pure projective in $\mathfrak{L}$ if and only if $G \cong \mathbb{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbb{Z}$ where n is a non-negative integer and $\mathfrak{m}$ is a cardinal.*

Proof. The statements follow from Theorem 2.8 and the proof of (ii) $\not\Rightarrow$ (i) of Theorem 2.7.

Let G be a group in $\mathfrak{L}$. Then G is said to be an *elementary p-group* if $pG = 0$ for some prime p. The group G is called (*topologically*) *pure-simple* if G contains no non-trivial closed (topologically) pure subgroups. Similarly, G is called (*topologically*) *pure-full* if every closed subgroup of G is (topologically) pure. The structure of pure-simple and pure-full LCA groups was completely determined by Armacost [1]:

Theorem 2.10 (Armacost [1]). *A group in $\mathfrak{L}$ is pure-simple if and only if it is isomorphic to one of the following*: (a) *a discrete group of rank* 1; (b) *the group of p-adic integers*; (c) *the group of p-adic numbers*; (d) $\mathbb{R}$; *or* (e) *a compact connected group of dimension* 1.

Theorem 2.11 (Armacost [1]). *A group in $\mathfrak{L}$ is pure-full if and only if it is a local direct product of elementary p-groups with respect to compact open subgroups.*

Both results remain valid if 'pure' is replaced by 'topologically pure':

Corollary 2.12. *Let G be a group in $\mathfrak{L}$. Then G is topologically pure-simple if and only if it is pure-simple, and G is topologically pure-full if and only if it is pure-full.*

Proof. Suppose that G is topologically pure-simple. Then $\hat{G}$ is pure-simple, and hence isomorphic to one of the groups (a)–(e) of Theorem 2.10, and so is G (see [2, p. 20] and [5, Theorems 24.25, 24.28]). It follows that G is pure-simple. Conversely, suppose that G is pure-simple. Then Theorem 2.10 shows that purity and topological purity are equivalent concepts in G, and thus G is topologically pure-simple. The second assertion follows immediately from Armacost's proof of Theorem 2.11 (see [1]).

3 Almost pure subgroups

A subgroup H of a group G is called *almost pure* if $nG \cap H \subseteq \overline{nH}$ for all positive integers n.

Lemma 3.1. *Let H be a subgroup of a group G. Then the following conditions are equivalent*:

(i) *H is almost pure in G*;

(ii) *$\overline{nG \cap H} = \overline{nH}$ for all n*;

(iii) *$\overline{n(n^{-1}H)} = \overline{nH}$ for all n*;

(iv) *$n^{-1}H \subseteq n^{-1}(\overline{nH})$ for all n.*

Proof. Clearly (i) implies (ii). Let n be a positive integer. Assuming (ii), we let $x \in n(n^{-1}H)$. Then $x = ny$ for some $y \in n^{-1}H$, so that

$$x = ny \in nG \cap H \subseteq \overline{nH}$$

and (iii) follows. Now assume (iii) and let $x \in n^{-1}H$. By our assumption, $nx \in \overline{nH}$, so that $x \in n^{-1}(\overline{nH})$, and (iv) follows. Finally, suppose that (iv) holds and let $x \in nG \cap H$. Then $x = ny$ for some $y \in n^{-1}(\overline{nH})$, so that $x = ny \in \overline{nH}$. Consequently, (iv) implies (i).

Note that the annihilator of a closed (topologically) pure subgroup of a group $G \in \mathfrak{L}$ need not be (topologically) pure in $\hat{G}$ (see for instance [7]). However, we have the following result:

Proposition 3.2. *Let H be a closed subgroup of a group G in $\mathfrak{L}$. Then H is almost pure in G if and only if $(\hat{G}, H)$ is almost pure in $\hat{G}$.*

Proof. Suppose that H is almost pure in G and let n be a positive integer. By Lemma 3.1, we have

$$\overline{n(n^{-1}H)} = \overline{nH},$$

and hence

$$n^{-1}(\hat{G}, H) = (\hat{G}, \overline{nH}) = (\hat{G}, \overline{n(n^{-1}H)}) = n^{-1}(\hat{G}, n^{-1}H) = n^{-1}(\overline{n(\hat{G}, H)}).$$

We apply Lemma 3.1 again and conclude that $(\hat{G}, H)$ is almost pure in $\hat{G}$. The converse follows from [5, Theorem 24.10].

A proper short exact sequence $0 \to A \xrightarrow{\phi} B \to C \to 0$ in $\mathfrak{L}$ is called *almost pure* if $\phi(A)$ is almost pure in B. It is clear that an extension equivalent to an almost pure extension is almost pure.

Lemma 3.3. *A pullback of an almost pure exact sequence is almost pure. A pushout of an almost pure exact sequence is almost pure.*

Proof. The proof of the first statement is similar to the first part of the proof for Lemma 2.2. Assume that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \theta} & & \big\| & & \\ 0 & \longrightarrow & G & \xrightarrow{\mu} & X & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is a standard pushout diagram in $\mathfrak{L}$, with top row an almost pure exact sequence. Then $X = (G \oplus B)/N$ where $N = \{(-f(a), \phi(a)) : a \in A\}$, $\mu : g \mapsto (g, 0) + N$ and $\theta : b \mapsto (0, b) + N$. Let n be a positive integer and let $n(g, b) + N \in nX \cap \mu(G)$. Then there are elements $a \in A$ and $g' \in G$ such that $n(g, b) = (g', 0) + (-f(a), \phi(a))$, and hence

$$\phi(a) \in nB \cap \phi(A) \subseteq \overline{n\phi(A)}.$$

For $x \in A$ we have

$$(0, n\phi(x)) = (nf(x), 0) + (-f(nx), \phi(nx)),$$

so that $n(g, b) \in \overline{(nG \oplus \{0\}) + N}$ and we conclude that $n(g, b) + N \in \overline{n\mu(G)}$. Consequently, $\mu(G)$ is almost pure in X.

The elements of $\mathrm{Ext}(C, A)$ represented by almost pure exact sequences form a subgroup denoted by $\mathrm{Apext}(C, A)$. By Proposition 3.2, this group is isomorphic to $\mathrm{Apext}(\hat{A}, \hat{C})$. A group in $\mathfrak{L}$ is called (*almost*) *pure injective in* $\mathfrak{L}$ if it has the injective property relative to the class of (almost) pure exact sequences. The following result can be found in [8].

Theorem 3.4. *An LCA group which is pure injective in $\mathfrak{L}$ possesses a dense divisible subgroup.*

Corollary 3.5. *For a group $G \in \mathfrak{L}$ the following conditions are equivalent*:

(i) *G is almost pure injective in $\mathfrak{L}$*;

(ii) $\mathrm{Apext}(X, G) = 0$ *for all groups X in $\mathfrak{L}$*;

(iii) *$G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$ where n is a non-negative integer and $\mathfrak{m}$ is a cardinal.*

Proof. A subgroup containing a dense divisible subgroup is almost pure, and so by Theorem 3.4 the group G is almost pure injective in $\mathfrak{L}$ exactly if it is injective in $\mathfrak{L}$. Since G is injective in $\mathfrak{L}$ if and only if $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$, the proof is complete.

Finally, we call a group in $\mathfrak{L}$ *almost pure projective in* $\mathfrak{L}$ if it has the projective property relative to the class of almost pure exact sequences. Then dualization of Corollary 3.5 yields

Corollary 3.6. *For a group* $G \in \mathfrak{L}$ *the following conditions are equivalent*:

(i) *G is almost pure projective in* $\mathfrak{L}$;

(ii) $\mathrm{Apext}(G, X) = 0$ *for all groups* X *in* $\mathfrak{L}$;

(iii) $G \cong \mathbb{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbb{Z}$ *where n is a non-negative integer and* $\mathfrak{m}$ *is a cardinal.*

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