4-2018

Number Theory: Niven Numbers, Factorial Triangle, and Erdos' Conjecture

Sarah Riccio
Sacred Heart University, riccios@mail.sacredheart.edu

Sacred Heart University

Mathematics Senior Seminar

MA 398

Number Theory

Niven Numbers, Factorial Triangle, and Erdos' Conjecture

Supervisor:

Dr. Bernadette Boyle

Author:

Sarah Riccio

December 8, 2017

Contents

Abstract

In this paper, three topics in number theory will be explored: Niven Numbers, the Factorial Triangle, and Erdos's Conjecture . For each of these topics, the goal is for us to find patters within the numbers which help us determine all possible values in each category. We will look at two digit Niven Numbers and the set that they belong to, the alternating summation of the rows of the Factorial Triangle, and the unit fractions whose sum is the basis of Erdos' Conjecture.

1 Introduction: Niven Numbers

In this section of the paper we will explore Niven Numbers and the patterns that they follow. A Niven Number is classified as a number which, when its digits are added together, the sum of the digits can divide the number with a remainder of zero. We will try to find if there is a set which all Niven Numbers can be classified into. For instance, all two digit Niven numbers can be considered to be part of the set $3\mathbb{Z} \cup 10\mathbb{Z}$. Further questions that will be explored are: Is there an infinite amount of Niven Numbers and are all Niven Numbers elements of $3\mathbb{Z} \cup 10\mathbb{Z}$?

1.1 Two Digit Niven Numbers

In this section, we want to explore what set we can classify two-digit Niven Numbers to be in.

Lemma 1.1. *If a positive, two-digit integer, n, is a Niven Number, then it is an element of the set $3\mathbb{Z} \cup 10\mathbb{Z}$.*

Proof. Let n be a two-digit, positive, integer which is a Niven Number, thus it is divisible by the sum of its digits. Let n be represented by st, where s is the integer in the tens place and t is the integer in the ones place, so that $(s+t)|n$.

First we must prove that $n \in 10\mathbb{Z}$. We will assume $n \notin 3\mathbb{Z}$. Since $n \notin 3\mathbb{Z}$, $s + t \notin 3\mathbb{Z}$ because $(s+t)|n$. Note that $0 < s \leq 9$ and $0 \leq t \leq 9$ because they represent digits of a number. Therefore $s + t$ can possibly equal:

$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17$

Case 1: We will look at the case where $n \in 10\mathbb{Z}$ and $s + t$ is even.

First we will look at the instance where $s+t$ is even. Then the sum of s and t can possibly be 2,4,8,10,14,16. Since $s+t$ is even, either both s and t are even or both s and t are odd. If both s and t are even, then $s+t$ must be an even number. Then the possibilities for n with those restrictions are: $20, 22, 28, 40, 46, 64, 68, 80, 82, 86, 88$. Out of the given choices, only $20, 40$, and 80 also fit the rule of Niven Numbers where the digits' sum divides the actual number. If s and t are both even, then n can equal: $20, 40, 80$.

Now we have to look at the instance where s and t are both odd. There are no numbers which fit this criterion. This is because if s and t are both odd, then n would be odd as well. But $s + t$ would be even, since the sum of two odd numbers is even. The number itself would not be divisible by the summation of its digits because an odd number can not be evenly divided by an even number. Thus, if s and t are both odd, then no Niven Number can be found.

Case 2: We will look at the case where $n \in 10\mathbb{Z}$ and $s + t$ is odd.

Since $s + t$ is odd we know that $s + t$ can equal: $1, 5, 7, 11, 13, 17$. We also know that since $s+t$ is odd, then one element of n is odd, and the other element is even.

First we will look at the instance where s is odd and t is even. Since t is even, then n must also be even. Since we know that n is even, the possibilities for n with those restrictions are: $10, 14, 16, 32, 34, 38, 50, 52, 56, 58, 70, 74, 76, 92, 94, 98$. Out of the given choices, only $10, 50$, and 70 fit the criteria of Niven Numbers where the digits' sum divides the actual number. Then n can equal $10, 50$, or 70.
Now we must look at the instance where $s + t$ is odd, where s is even and t is odd. This would mean that n would also be odd. This can not occur because no multiple of ten is an odd number, therefore contradicting our original claim.

Since the only numbers not in $3\mathbb{Z}$ which are Niven Numbers that n can equal are $10, 20, 40, 50, 70, 80$, $n \in 10\mathbb{Z}$.

Now we need to show that if a Niven Number, n is not in $10\mathbb{Z}$, then $n \in 3\mathbb{Z}$. Assume $n \notin 10\mathbb{Z}$, then $(s + t) \notin 10\mathbb{Z}$, since $(s + t)|n$. Note that $0 < s \leq 9$ and $0 \leq t \leq 9$ because s and t represent the digits of the Niven Number, n. Therefore $s + t$ can possibly equal $1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17$, or 18.

Case 1: We will look at the case where $n \in 3\mathbb{Z}$ and $s + t$ is even.
If $s+t$ is even, then the sum can be : $2, 4, 6, 8, 12, 14, 16$, or 18. Since $s+t$ is even, we know that either both s and t are even or both s and t are odd. First we will look at the instance where s and t are both even. If s and t are both even, then n

must also be even. Then n can equal $24, 42$, or 84. Using the same process as seen earlier in the proof, it can clearly be seen that $24, 42$, and 84 are the only two-digit even numbers which are also Niven Numbers.

Now we must look at the instance where both s and t are odd. If s and t are both odd, $s + t$ would not be able to divide n because n would be odd, but $s + t$ would be even, since the sum of two odd integers is an even integer. An odd integer cannot be evenly divided by an even number, therefore s and t cannot both be odd.

Case 2: Now we will look at the case where $s + t$ is odd.

If $s+t$ is odd, then the sum can be: $1, 3, 5, 7, 9, 11, 13, 15$, or 17. Then one element is odd and the other is even. First we will look at the instance where s is odd and t is even. Then n is even and can equal $12, 18, 36, 54$, or 72. Using the same process as seen earlier in the proof, it can be seen that the only two-digit, even numbers which also are Niven Numbers are $12, 18, 36, 54, 72$.

Now we must look at the case where s is even and t is odd. Then n is odd.Then n can equal $21, 27, 45, 63$, or 81.Using the same process as seen earlier in the proof, it can be seen that the only two-digit, numbers which also are Niven Numbers are $21, 27, 45, 63, 81$. Now we have shown that if $n \notin 10\mathbb{Z}$ it is the case that $n \in 3\mathbb{Z}$.

Finally we must look at the case where $n \in 10\mathbb{Z}$ and $n \in 3\mathbb{Z}$. If n is in both $3\mathbb{Z}$ and $10\mathbb{Z}$, then n must be in $30\mathbb{Z}$. The only two-digit numbers which are in

$30\mathbb{Z}$ are $30, 60$, and 90. It can clearly be seen that the sum of their digits divides the number, n, therefore any $n \in 30\mathbb{Z}$ that is a two digit number must be a Niven Number.

Now that we have shown that $n \in 3\mathbb{Z}$, $n \in 10\mathbb{Z}$, or $n \in 30\mathbb{Z}$ we know that if a positive, two-digit integer, n, is a Niven Number, then $n \in 3\mathbb{Z} \cup 10\mathbb{Z}$.

□

1.2 Further Questions on Niven Numbers

Some further questions that we could research on Niven numbers is when n is guaranteed not to be a Niven Number, if there are infinitely many Niven Numbers, and if all Niven Numbers are solely elements of $3\mathbb{Z} \cup 10\mathbb{Z}$.

2 Factorial Triangle

In this section of the paper we will explore the Fancy Triangle and the patterns that the numbers which make the triangle up follow. By exploring the patterns that the triangle follows, we then have the possibility to expand the triangle infinitely. The triangle is given below.

row 1				1		
row 2			1		1	
row 3		1		3		2
row 4		1	6		11	6
row 5	1	10		35	50	24
row 6	1	15	85	225	274	120

$\vdots$

The given general expansion of the triangle is given as the equation

$$A_{(i,j)} = (i-1) * A_{(i-1,j-1)} + A_{(i-1,j)} \tag{1}$$

Where i is the row number starting at 1, and j is the entry number counting from the right, starting at 1. Using this formula we can both expand the triangle, and use it to prove other mathematical patterns within the triangle.

2.1 Patterns Within Triangle

Looking at the rows in this triangle, we can see that if the alternating sum of a row is taken, the alternating sum is always equal to zero for every row greater than one.

Lemma 2.1. *For every row greater than one,*

$$0 = \sum_{j=1}^{i} (-1)^{(j+1)} A_{(i,j)} \tag{2}$$

Proof. To prove that the alternating sum of each row is equal to zero, we will use a proof by induction. First we will look at the base case, row 2. It is clear that when $n = 2$

$$0 = \sum_{j=1}^{2} (-1)^{(j+1)} A_{(2,j)} \tag{3}$$

This is because $1 + (-1) = 0$. Since our base case is true, we can now assume it is true for row n. So for row n, we have

$$0 = \sum_{j=1}^{n} (-1)^{(j+1)} A_{(n,j)} = A_{(n,1)} - A_{(n,2)} + ... \pm A_{(n,n)} \tag{4}$$

Now that we have looked at row n, we want to prove that the alternating summation of row $n + 1$ is also equal to 0. First we have

$$0 = \sum_{j=1}^{n+1} (-1)^{(j+1)} A_{(n+1,j)} = A_{(n+1,1)} - A_{(n+1,2)} + ... \pm A_{(n+1,n+1)} \tag{5}$$

We can rewrite the terms for row $n + 1$ in terms of n using equation one which gives us the general expansion of the triangle. Now we can say that

$$0 = \sum_{j=1}^{n+1} (-1)^{(j+1)} A_{(n+1,j)} = A_{(n,1)} - ((n)A_{(n,1)} + A_{(n,2)} + ... \pm (n)A_{(n,n)} + A_{(n,n+1)} \tag{6}$$

Continuing in this manner, we can simplify the equation further. Recall that our j values can not exceed our i values, and that our j values must be greater than zero, causing some of the terms in our expanded notation to not exist, and therefore not be used. We can now say that:

$$0 = \sum_{j=1}^{n+1} (-1)^{(j+1)} A_{(n+1,j)} = (n-1)[-A_{(n,1)} + A_{(n,2)} - ... \pm A_{(n,n)}] \tag{7}$$

Since we know that the alternating summation of row n is equal to zero, we then know that te alternating summation of row $n+1$ must also equal zero since $(n-1)(0) = 0$. Furthermore, we know that the terms in each row are what cause the alternating summation of the row to be zero, not the coefficient. This is because the smallest number that n can be is 2, and $2 - 1 = 1$, meaning the least value the coefficient can reach is 1, not zero. This shows that the alternating summation of the row is zero due to the entries in that row. □

3 Erdos's Conjecture

Erdos's Conjecture states that if n is an integer greater than 4, then $\frac{4}{n}$ is the sum of three or fewer distinct unit fractions. In this section we will explore if there is a way to tell if $\frac{4}{n}$ will be the sum of one, two, or three distinct unit fractions, and what those unit fractions are specifically. Erdos' Conjecture has not yet been fully proven for all numbers.

3.1 How to Find Unit Fractions

Given an integer $n \geq 4$, we will find the unit fractions whose sum is the fraction $\frac{4}{n}$. By determining the remainder of the given integer, n, when divided by four, we can find how many unit fractions are needed to sum to $\frac{4}{n}$, as well as what the fractions are. We will begin this process looking at the unit fractions that make up $\frac{4}{n}$ when $n \in 4\mathbb{Z}$, and then continue onwards for the cases of $n \in 4\mathbb{Z}+1$,$n \in 4\mathbb{Z}+2$, and $n \in 4\mathbb{Z}+3$.

Lemma 3.1. *If $n \in 4\mathbb{Z}$, then $\frac{4}{n}$ can be written as one unit fraction, namely $\frac{1}{k}$ where $k = \frac{n}{4}$.*

Proof. Let $n = 4k$ where $k \in \mathbb{Z}$ and $k > 1$

Then $\frac{4}{n} = \frac{4}{4k} = \frac{1}{k}$

$\frac{1}{k}$ is a unit fraction, therefore completing Erdos's Conjecture with one unit fraction.

□

Lemma 3.2. *For every integer, n where $n \in 4\mathbb{Z}+1$, and $n+3 \in 3\mathbb{Z}$, the fraction $\frac{4}{n}$ can be written as two distinct unit fractions, specifically $\frac{1}{k} + \frac{1}{j(12j-3)} = \frac{4}{n}$ where $k = \frac{n+3}{4}$ and $j = \frac{n+3}{12}$.*

Proof. Given $n \in 4\mathbb{Z}+1$, where $s-4 < n < s$, and n is greater than 4, let $s \in 4\mathbb{Z}$, where s is greater than 4. Since $n \in 4\mathbb{Z}+1$ we can say that $s-4 < n < s$. It follows that $n = s-3$. Also let $s = 4k$ where $k \in \mathbb{Z}$ and k is greater than 1. Since

$n < s$, we can subtract $\frac{4}{s}$ from $\frac{4}{n}$;

$\frac{4}{n} - \frac{4}{s} = \frac{4}{s-3} - \frac{4}{s} = \frac{4s-4(s-3)}{s(s-3)} = \frac{4s-4s+12}{s(s-3)} = \frac{12}{s^2-3s}$.

Since $n \in 3\mathbb{Z}$, and $n = s-3$, then $s \in 3\mathbb{Z}$. Since $s \in 3\mathbb{Z}$, and $s = 4k$, we can write $s = 12j$ where $j \in \mathbb{Z}$ and $3j = k$. We can continue using substitution: $\frac{12}{12j(12j-3)} = \frac{1}{j(12j-3)}$.

We then know that any number, n, where $n \in 4\mathbb{Z}+1$, and also $n \in 3\mathbb{Z}$ is made up of two unit fractions, namely, $\frac{4}{n} = \frac{1}{k} + \frac{1}{j(12j-3)}$ □

Lemma 3.3. *For every integer, n where $n \in 4\mathbb{Z}+1$, and also $n \notin 3\mathbb{Z}$, there exists $s \in 4\mathbb{Z}$, where $s-4 < n < s$, where $\frac{4}{n}$ can be written as three distinct unit fractions, specifically $\frac{1}{k} + \frac{1}{nk} + \frac{1}{j} = \frac{4}{n}$ where $k = \frac{n}{4}$ is even and $j = \frac{nk}{2}$.*

Proof. Given $n \in 4\mathbb{Z}+1$ and n is greater than 4, let $s \in 4\mathbb{Z}$ where s is greater than 4. Since $n \in 4\mathbb{Z}+1$ and $s \in 4\mathbb{Z}$, we can say that $s-4 < n < s$. It follows that $n = s-3$. Also let $s = 4k$ where $k \in \mathbb{Z}$ and k is greater than 1.Since $n < s$, we know that we can subtract $\frac{4}{s}$ from $\frac{4}{n}$;

$\frac{4}{n} - \frac{4}{s} = \frac{4}{s-3} - \frac{4}{s} = \frac{4s-4(s-3)}{s(s-3)} = \frac{4s-4s+12}{s(s-3)} = \frac{12}{s^2-3s}$.

Using Substitution where $s = 4k$: $\frac{12}{16k^2-12k} = \frac{3}{k(4k-3)}$.

Consider that k is even. We know that $4k-3 = n$, so we can rewrite $\frac{3}{k(4k-3)}$ as $\frac{4}{n} - \frac{1}{k} = \frac{3}{nk}$. Since we know that k is even, we know that nk is also even. We can write $\frac{3}{nk}$ as $\frac{1}{nk} + \frac{2}{nk}$. Since we know that nk is even, we let $nk = 2j$ where $j \in \mathbb{Z}$.Then, using substitution, $\frac{4}{n} - \frac{1}{k} = \frac{1}{nk} + \frac{2}{2j} = \frac{1}{nk} + \frac{1}{j}$. We then know that any number, n, where $n \in 4\mathbb{Z}+1$, and also $n \notin 3\mathbb{Z}$ is made up of three unit fractions, namely, $\frac{4}{n} = \frac{1}{k} + \frac{1}{j} + \frac{1}{nk}$ when $k = \frac{n+3}{4}$ and $j = \frac{n^2+3n}{8}$.

□

Lemma 3.4. *For every integer, n where $n \in 4\mathbb{Z}+2, \frac{4}{n}$ can be written as two distinct unit fractions, specifically $\frac{1}{k} + \frac{1}{2k^2-k} = \frac{4}{n}$, where $k = \frac{n+2}{4}$.*

Proof. Given $n \in 4\mathbb{Z}+2$ and n is greater than 4, let $s \in 4\mathbb{Z}$ where s is greater than 4. Since $n \in 4\mathbb{Z}+2$ and $s \in 4\mathbb{Z}$, we can say that $s-4 < n < s$. It follows that $n = s-2$. Also let $s = 4k$ where $k \in \mathbb{Z}$ and k is greater than 1.Since $n < s$, we know that we can subtract $\frac{4}{s}$ from $\frac{4}{n}$.

This gives us: $\frac{4}{n} - \frac{4}{s} = \frac{4}{s-2} - \frac{4}{s} = \frac{4s-4(s-2)}{s(s-2)} = \frac{4s-4s+8}{s(s-2)} = \frac{8}{s^2-2s}$.

Then, since we know that $s = 4k$, we have, $\frac{8}{(4k)^2-8k} = \frac{8}{16k^2-8k} = \frac{1}{2k^2-k}$.

Since $\frac{1}{2k^2-k}$ is a unit fraction, we then can write $\frac{4}{n}$ as $\frac{4}{n} = \frac{1}{k} + \frac{1}{2k^2-k}$ □

Lemma 3.5. *For every integer, n where $n \in 4\mathbb{Z}+3, \frac{4}{n}$ can be written as two distinct unit fractions, specifically $\frac{1}{k} + \frac{1}{4k^2-k} = \frac{4}{n}$, where $k = \frac{n+1}{4}$.*

Proof. Given $n \in 4\mathbb{Z}+3$ and n is greater than 4, let $s \in 4\mathbb{Z}$ where s is greater than 4. Since $n \in 4\mathbb{Z}+3$ and $s \in 4\mathbb{Z}$, we can say that $s-4 < n < s$. It follows that $n = s-1$. Also let $s = 4k$ where $k \in \mathbb{Z}$ and k is greater than 1.Since $n < s$, we know that we can subtract $\frac{4}{s}$ from $\frac{4}{n}$.

This gives us: $\frac{4}{n} - \frac{4}{s} = \frac{4}{s-1} - \frac{4}{s} = \frac{4s - 4(s-1)}{s(s-1)} = \frac{4s - 4s + 4}{s(s-1)} = \frac{4}{s^2 - s}$. Then, since we know that $s = 4k$, we have, $\frac{4}{(4k)^2 - 4k} = \frac{4}{16k^2 - 4k} = \frac{1}{4k^2 - k}$. Since $\frac{1}{4k^2 - k}$ is a unit fraction, we then can write $\frac{4}{n}$ as $\frac{4}{n} = \frac{1}{k} + \frac{1}{4k^2 - k}$ $\square$

3.2 Further questions on Erdos' Conjecture

Since Erdos' Conjecture has not been fully proven, it would be interesting to investigate the case where $n \in 4\mathbb{Z}+1$ and k, which is equal to $\frac{n}{4}$ is an odd number which is not in $3\mathbb{Z}$. This would complete the proof of Erdos' Conjecture.

4 Conclusion

In conclusion we have looked through Niven Numbers, the Factorial Triangle, and Erdos' Conjecture. All three topics have the ability to be expanded upon in further research. Erdos' Conjecture poses a very interesting portion of further research specifically in integers which belong to $4\mathbb{Z} + 1$.

5 References

Schwartzman, Steven. (November, 1984) The Factorial Triangle and Polynomial Sequences. Retrieved from https://www.maa.org/sites/default/files/0746834202308.di020714-.02p0042n.pdf. Last accessed on Dec 04, 2017.