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# A Density Property of the Tori and Duality

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## A Density Property of the Tori and Duality.

PETER LOTH(\*)

ABSTRACT - In this note, a short proof of a recent theorem of D. Dikranjan and M. Tkachenko is given, and their result is extended.

### 1. Introduction.

Let us establish notation and terminology. The additive groups of integers and rational numbers taken discrete are denoted by  $\mathbf{Z}$  and  $\mathbf{Q}$  respectively. Recall that two nontrivial subgroups of  $\mathbf{Q}$  are isomorphic if and only if they are of the same type (see [F] Theorem 85.1). The type of  $\mathbf{Z}$  is written  $\mathbf{t}_0$ . By  $\mathbf{R}$  we mean the additive topological group of real numbers, and by  $\mathbf{T}$  we denote the one-dimensional torus with its usual topology. All considered groups are locally compact abelian groups and will be written additively. The subgroup of all compact elements of the group  $G$  is denoted by  $bG$ , while  $G_0$  stands for the identity component of  $G$ . The Pontrjagin dual of  $G$  is denoted by  $\widehat{G}$ . If  $H$  is a subset of  $G$ , then  $(\widehat{G}, H)$  is the annihilator of  $H$  in  $\widehat{G}$ . Throughout this paper, we use the term «isomorphic» for «topologically isomorphic». For details and fundamental results on locally compact abelian groups and Pontrjagin duality, we may refer to the book [HR] of Hewitt and Ross.

For a group  $G$ ,  $a \in G$ , and a positive integer  $n$  let

$$S_n(a) = \{x \in G : nx = a\}.$$

Notice that if  $G$  is divisible, in particular if it is a compact connected group,  $S_n(a)$  is nonempty. Following Dikranjan and Tkachenko [DT], we

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say that a compact metrizable group  $G$  with an invariant metric  $d$  satisfies *condition  $\varepsilon$*  if for every  $a \in G$  and for every positive integer  $n$  there is an  $\varepsilon_n > 0$  such that  $d(y, S_n(a)) \leq \varepsilon_n$  for every  $y \in G$  and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . In [DT], it was shown that the finite-dimensional tori are exactly those compact connected abelian groups having the property that every closed connected one-dimensional subgroup satisfies condition  $\varepsilon$  (see [DT] Theorem 3.1). In this note, we give a short proof of a more general result.

Pontrjagin duality shows that a nontrivial compact metrizable group satisfies condition  $\varepsilon$  if and only if its dual group is homogeneous of type  $t_0$  (Lemma 2.1). A locally compact abelian group  $G$  has the form  $\mathbf{R}^n \times \mathbf{Z}^m \times bG$  exactly if every discrete torsion-free rank-one group  $G/H$  is cyclic (see Theorem 2.2). This yields a characterization of finite rank free abelian groups among the class of all discrete torsion-free groups (Corollary 2.3). The groups of the form  $\mathbf{R}^n \times \mathbf{T}^m \times D$  (where  $D$  is totally disconnected) are precisely those locally compact abelian groups having the property that every compact connected one-dimensional subgroup satisfies condition  $\varepsilon$  (see Theorem 2.4). As an immediate consequence, [DT] Theorem 3.1 follows.

## 2. Condition $\varepsilon$ and duality.

The following preliminary lemma will be useful:

LEMMA 2.1. *A nontrivial compact metrizable abelian group satisfies condition  $\varepsilon$  if and only if its discrete dual group is homogeneous of type  $t_0$ .*

PROOF. Suppose  $G \neq 0$  is a compact metrizable group. By [DT] Lemma 2.4,  $G$  satisfies condition  $\varepsilon$  exactly if for every prime  $p$  and every infinite set  $\pi$  of primes, the  $p$ -torsion part  $t_p(G)$  and the  $\pi$ -socle  $\text{soc}_\pi(G) = \bigoplus_{q \in \pi} G[q]$  are dense subgroups of  $G$ . Since

$$(\widehat{G}, \overline{t_p(G)}) = p^\omega \widehat{G} \quad \text{and} \quad (\widehat{G}, \overline{\text{soc}_\pi(G)}) = \bigcap_{q \in \pi} q \widehat{G},$$

$G$  satisfies condition  $\varepsilon$  if and only if every nonzero element of  $\widehat{G}$  is divisible by only finitely many primes and has finite  $p$ -height for every prime  $p$ , i.e., if  $\widehat{G}$  is homogeneous of type  $t_0$ . ■

**THEOREM 2.2.** *The following conditions are equivalent for a locally compact abelian group  $G$ :*

- (i) *Every discrete torsion-free rank-one group  $G/H$  is cyclic;*
- (ii)  *$G$  is isomorphic to  $\mathbf{R}^n \times \mathbf{Z}^m \times bG$  where  $n$  and  $m$  are non-negative integers;*
- (iii) *every nontrivial discrete torsion-free group  $G/H$  is homogeneous of type  $\mathbf{t}_0$ .*

**PROOF.** (i) implies (ii). Suppose  $G$  satisfies condition (i). Let  $G' = G/(G_0 + bG)$  and note that  $G'$  is discrete. Any locally compact abelian group is isomorphic to  $\mathbf{R}^n \times L$  with a nonnegative integer  $n$  and  $L$  containing a compact open subgroup (see [HR] Theorem 24.30); thus  $G' = G/(G_0 + bG) \cong (\mathbf{R}^n \times L)/(\mathbf{R}^n \times bL) \cong L/bL$  is torsion-free. If  $G'$  has infinite rank, it contains a free subgroup  $Z$  of infinite rank. Consequently,  $\mathbf{Q}$  is a homomorphic image of  $Z$ . Since  $\mathbf{Q}$  is injective, it is isomorphic to a factor group of  $G$ , a contradiction. Therefore  $G'$  has finite rank.

If  $\text{rank}(G') = 1$ , then  $G'$  is isomorphic to  $\mathbf{Z}$ , hence  $G$  is isomorphic to  $\mathbf{R}^n \times \mathbf{Z} \times bG$ . If  $\text{rank}(G') = k > 1$ ,  $G'$  contains a free subgroup  $F$  of rank  $k$ , say,  $F = X_1 \oplus \dots \oplus X_k$  where each group  $X_i$  is isomorphic to  $\mathbf{Z}$ . Let  $F_i = \bigoplus_{j \neq i} X_j$  and  $Y_i$  the purification of  $F_i$  in  $G'$ . Then we have an embedding

$$G' \rightarrow G'/Y_1 \oplus \dots \oplus G'/Y_k,$$

and every group  $G'/Y_i$  is isomorphic to  $\mathbf{Z}$ . Consequently,  $G'$  is free and therefore  $G$  has the form  $\mathbf{R}^n \times \mathbf{Z}^m \times bG$ , as desired.

(ii) implies (iii). Suppose  $G$  satisfies (ii). If the quotient  $G/H$  is discrete and torsion-free, then  $bG + G_0$  is a subgroup of  $H$ , hence  $G/H$  is finitely generated and (iii) follows.

By [F] Theorem 85.1, (iii) implies (i). ■

**COROLLARY 2.3.** *The following conditions are equivalent for a discrete torsion-free abelian group  $G$ :*

- (i) *Every torsion-free rank-one group  $G/H$  is cyclic;*
- (ii)  *$G$  is a finite rank free abelian group;*
- (iii) *every nontrivial torsion-free group  $G/H$  is homogeneous of type  $\mathbf{t}_0$ .*

Locally compact abelian groups whose identity component have the form  $\mathbf{R}^n \times \mathbf{T}^m$ , can be characterized in terms of condition  $\varepsilon$ :

**THEOREM 2.4.** *The following conditions are equivalent for a locally compact abelian group  $G$ :*

- (i) *Every compact connected one-dimensional subgroup of  $G$  satisfies condition  $\varepsilon$ ;*
- (ii)  *$G$  is isomorphic to  $\mathbf{R}^n \times \mathbf{T}^m \times D$  where  $n$  and  $m$  are nonnegative integers, and  $D$  is totally disconnected;*
- (iii) *every nontrivial compact connected subgroup  $L$  of  $G$  is metrizable and  $\widehat{L}$  is homogeneous of type  $\mathfrak{t}_0$ ;*
- (iv) *every compact connected subgroup of  $G$  is metrizable and satisfies condition  $\varepsilon$ .*

**PROOF.** Using Pontrjagin duality (see [HR] Theorems 24.25, 24.28) (i) implies (ii) because of Lemma 2.1 and Theorem 2.2. Suppose  $G$  is a group as in (ii) and  $L$  is a compact connected subgroup of  $G$ . Then  $L$  is a torus, hence  $\widehat{L}$  is a free abelian group and therefore (ii) implies (iii). By Lemma 2.1, (iii) implies (iv). Clearly (iv) implies (i). ■

As a corollary, we obtain

**THEOREM 2.5** (Dikranjan and Tkachenko [DT]). *The following conditions are equivalent for a compact connected abelian group  $G$ :*

- (i) *Every closed connected one-dimensional subgroup of  $G$  satisfies condition  $\varepsilon$ ;*
- (ii)  *$G$  is a finite-dimensional torus;*
- (iii) *every closed connected subgroup of  $G$  is metrizable and satisfies condition  $\varepsilon$ .*

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