2014

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Abstract

This paper provides an explicit construction of Kleinian groups that have small Hausdorff dimension of their limit sets. It is known that such groups exist and they can be constructed by results of Patterson. The purpose here is to work out the methods of calculation.

Keywords: Kleinian groups, Hausdorff dimension, exponent of convergence, Patterson-Sullivan measure
2000 MSC: 30F40, 37F35

1. Introduction

For the basics of Kleinian group theory, see [7], [8], or [9]. We denote the group of orientation preserving isometries of hyperbolic n-space by $\text{Isom}^+(\mathbb{B}^n)$. A *Kleinian group* Γ is a discrete subgroup of $\text{Isom}^+(\mathbb{B}^n)$. The *limit set* $\Lambda(\Gamma)$ of a Kleinian group Γ is the subset of $\partial\mathbb{B}^n$ defined for any $x \in \mathbb{B}^n$ by $\Lambda(\Gamma) = \overline{\Gamma(x)} \cap \partial\mathbb{B}^n$. A Kleinian group Γ is *elementary* when $\Lambda(\Gamma)$ contains at most two points. A Kleinian group whose limit set contains more than two points is called *non-elementary.* The complement of the limit set, $\Omega(\Gamma) = \partial\mathbb{B}^n - \Lambda(\Gamma)$ is the *domain of discontinuity.* If $\Omega(\Gamma) = \emptyset$ then Γ is a Kleinian group of the *first kind*, otherwise it is a Kleinian group of the *second kind.*

The group $\mathrm{Isom}^+(\mathbb{B}^n)$ has a natural identification with the group of conformal homeomorphisms on $\partial\mathbb{B}^n$. Given this connection, we now classify elements of a Kleinian group. An element γ in a Kleinian group Γ is called *elliptic* if it has a fixed point in $\mathbb{B}^n$. If it has exactly one fixed point in $\partial\mathbb{B}^n$ then it is called *parabolic*. All other elements are called *loxodromic*. Loxodromic elements have two fixed points in $\partial\mathbb{B}^n$ and they set-wise fix the hyperbolic line connecting them. This line is called the *axis* of the loxodromic element. For any point x on the axis, the group action of increasing positive powers of a loxodromic element γ moves x towards one of the fixed points of γ called the *attracting fixed point*. The other fixed point is called the *repelling fixed point*. A *torsion-free* Kleinian group contains no elliptic elements. If Γ is a torsion-free Kleinian group then the quotient $\mathbb{B}^n/\Gamma = M$ is a hyperbolic n-manifold. From this point forward we only consider torsion-free Kleinian groups.

The *conical limit set* $\Lambda_c(\Gamma)$ of a Kleinian group Γ acting on $\mathbb{B}^n$ is the set of points $\zeta \in \Lambda(\Gamma)$ where there exists a sub-orbit of $\Gamma(x)$ that converges to ζ within a bounded hyperbolic distance from a geodesic ray ending at ζ. Note that any loxodromic fixed point is a conical limit point.

For a Kleinian group Γ the *convex hull* of $\Lambda(\Gamma)$, which we denote by $CH(\Lambda(\Gamma))$, is the minimal convex subset of $\mathbb{B}^n$ that contains all geodesics with endpoints in $\Lambda(\Gamma)$. The *convex core* $C(\Gamma)$ of the associated hyperbolic manifold $M = \mathbb{B}^n/\Gamma$ is the quotient $CH(\Lambda(\Gamma))/\Gamma$. The convex core is the smallest convex sub-manifold of M so that the inclusion map is a homotopy equivalence.

We also consider another subset of a hyperbolic manifold. For $\varepsilon > 0$, define $M_{(0,\varepsilon]}$ to be the subset of points $p \in M$ such that there is a non-trivial closed curve passing through p with length less than or equal to ε. We call $M_{(0,\varepsilon]}$ the $\varepsilon-$*thin* part of M and the complement, $M_{[\varepsilon,\infty)}$ the $\varepsilon-$*thick* part. A Kleinian group Γ is *geometrically finite* if for any $\varepsilon > 0$, the convex core intersected with the $\varepsilon-$*thick* part of the associated manifold M is a compact set.

We will be working with a specific type of Kleinian group, a Schottky group. A *Schottky group* is a Kleinian group Γ generated by transformations $\gamma_1, \ldots, \gamma_n$, where there are $2n$ $(n \geq 2)$ disjoint Jordan curves, $C_1, C_1', \ldots, C_n, C_n'$, bounding a common domain D in $\widehat{\mathbb{C}}$ where $\gamma_i(C_i) = C_i'$ and $\gamma_i(D) \cap D = \emptyset$ for each i. The Jordan curves $C_1, C_1', \ldots, C_n, C_n'$ are called *defining loops* for the *Schottky generators* $\gamma_1, \ldots, \gamma_n$. A rank n Schottky group has n Schottky generators. If Γ is a rank r Schottky group then $(\mathbb{B}^3 \cup \Omega(\Gamma))/\Gamma$ is a handlebody with r handles.

The subgroups used in our construction are formed from rank two Schottky groups. We briefly outline how one may build such a group. Consider a Möbius transformation m acting on $\widehat{\mathbb{C}}$ where $m(\infty) \neq \infty$. Then the point $c = m^{-1}(\infty)$ is called the *center of the isometric circle* of m. Furthermore, we say $c' = m(\infty)$ is the center of the isometric circle of m^{-1}. Recall that m maps the family of Euclidean circles centered at c onto the family of circles centered at c'. There is a unique circle I in the first family which is mapped to a circle of the second family with the same radius; it is called the *isometric circle* of m and its image I' is the isometric circle of m^{-1}. Note that in $\mathbb{H}^3$ the spherical cap bounded by I is called the *isometric sphere*. Now let γ_1 and γ_2 be loxodromic transformations so that the four isometric circles (the isometric circles of both elements and their inverses) do not intersect and no circle is contained in another. Then $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is a rank two Schottky group.

As stated above, Schottky groups are used in our construction. Since they are our building blocks, we need more information about them. For a very detailed discussion please see [8, Chapter X].

Proposition 1.1. *[8, X.H.2] If Γ is a Schottky group on the generators $\gamma_1, \dots, \gamma_n$ acting on $\widehat{\mathbb{C}}$, then Γ is purely loxodromic, geometrically finite, and free on the generators $\gamma_1, \dots, \gamma_n$.*

Conversely, if we are given a Kleinian group with certain characteristics we may identify it as a Schottky group.

Theorem 1.2. *[8, X.H.6] If Γ is a Kleinian group of the second kind acting on $\widehat{\mathbb{C}}$, purely loxodromic, free, and finitely generated, then Γ is a Schottky group.*

In our construction we will consider the group $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ where Γ_1 and Γ_2 are Kleinian groups. We must assert that Γ is a Kleinian group and then we would like to compare the limit set of Γ with the limit sets of Γ_1 and Γ_2. In order for Γ to be a Kleinian group, Γ_1 and Γ_2 must satisfy certain criteria. The following is called Klein's combination theorem. We refer the reader to [8, Chapter VII] for more information on combining Kleinian groups.

Theorem 1.3. *[8, VII.A.13] Suppose Γ_1 and Γ_2 are Kleinian groups such that D_i is a fundamental set for Γ_i, $D_1 \cup D_2 = \partial \mathbb{B}^n$ and $D_1 \cap D_2 \neq \emptyset$. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle = \Gamma_1 * \Gamma_2$ is also a Kleinian group.*

We now define the Hausdorff dimension, a number which provides geometric properties of the limit set of a Kleinian group and information about the group itself (see [1], [7, Theorem 3.14.1]).

Let U be a subset of a metric space (X, ρ) and let $\operatorname{diam}(U)$ denote the diameter of U. Suppose E is a Borel subset of X and $\alpha \geq 0$. For $\varepsilon > 0$,

$$\mathcal{H}^\alpha_\varepsilon(E) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^\alpha : E \subseteq \bigcup_{i=1}^{\infty} U_i;\ \operatorname{diam}(U_i) \leq \varepsilon \right\}.$$

The *α-dimensional Hausdorff measure* of E is $\mathcal{H}^\alpha(E) = \lim_{\varepsilon \to 0} \mathcal{H}^\alpha_\varepsilon(E)$ and the *Hausdorff dimension* of a set E, denoted $\dim(E)$, is defined by

$$\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha : \mathcal{H}^\alpha(E) = +\infty\}.$$

For certain Kleinian groups, one may acquire information about the Hausdorff dimension of the limit set by using the critical exponent of the group. In order to define this exponent, we begin with the Poincaré series. For Γ a Kleinian group, $x, y \in (\mathbb{B}^n, d)$, and $s \geq 0$ we define the *Poincaré series*

$$P_s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma(y))}.$$

Theorem 1.4. *[10, Theorem 1.6.1] If $s > n-1$ (the dimension of $\partial\mathbb{B}^n$) then the Poincaré series converges.*

The *critical exponent* or *exponent of convergence* $\delta = \delta(\Gamma)$ of a Kleinian group Γ is

$$\delta = \inf \left\{ s : \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma(y))} < \infty \right\}.$$

If $P_\delta(x, y)$ converges then Γ is of *convergence type*. If $P_\delta(x, y)$ diverges then Γ is of *divergence type*.

Suppose that Γ is a Kleinian group with limit set $\Lambda(\Gamma)$. Here is a relationship between the critical exponent of Γ and the Hausdorff dimension of the conical limit set. Note that the following theorem first appeared in [1] by Bishop and Jones for Kleinian groups acting on $\mathbb{B}^3$. It was then generalized in [12] by Stratmann to higher dimensions.

Theorem 1.5. *If Γ is a non-elementary Kleinian group, then $\delta(\Gamma) = \dim(\Lambda_c(\Gamma))$.*

This yields the following immediate corollary.

Corollary 1.6. *If* Γ *is a non-elementary geometrically finite Kleinian group then* $\dim(\Lambda(\Gamma)) = \delta(\Gamma)$.

The final piece of information needed in our construction relates the group $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ to its generating groups. Given the last corollary, we will use the critical exponent to determine the Hausdorff dimension of the limit set of Γ. However, in order to do this, we need Γ to be geometrically finite. The following theorem provides a circumstance for this occasion. Note that this theorem appears in [8, VII.C.2 (xi)] for Kleinian groups acting on $\mathbb{B}^3$. It was generalized to higher dimensions in [5].

Theorem 1.7. *Suppose that* $\Gamma, \Gamma_1,$ *and* Γ_2 *are groups as in Theorem 1.3. Then* Γ *is geometrically finite if and only if* Γ_1 *and* Γ_2 *are both geometrically finite.*

The preceding concepts are sufficient to understand the construction offered here. Let Γ be a non-elementary Kleinian group. In this paper we consider the Hausdorff dimension of limit sets of subgroups of Γ. We will explicitly construct a Kleinian group Γ_ε so that for a given $\varepsilon > 0$, the Hausdorff dimension of the limit set of Γ_ε is less than ε. The following result of Patterson implies the existence of such a construction.

Theorem 1.8. *[11, Theorem 1] Suppose* $d_E(S_1, S_2)$ *denotes the Euclidean distance between two subsets* S_1 *and* S_2 *of* $\mathbb{B}^n$, $H = \{h_i : i \in I\} \subseteq Isom^+(\mathbb{B}^n)$, *and* Γ_1, Γ_2 *are Kleinian groups with convex fundamental polyhedrons* F_1, F_2 *respectively such that* $F_1^c \cap h(F_2^c) = \emptyset$ $(h \in H)$ *and* $\gamma(F_i) \cap F_j = \emptyset$ $(\gamma \in \Gamma_i - \{id\})$. *If* $h_i \in H$ *and*

$$\frac{\sup_{w \in F_2^c} \left(\frac{1-|h_i(w)|^2}{1-|w|^2} \right)}{d_E(F_1^c, h_i(F_2^c))} \to 0$$

as $i \to \infty$, *then* $\delta(\Gamma_1 * (h_i \Gamma_2 h_i^{-1})) \to \max\{\delta(\Gamma_1), \delta(\Gamma_2)\}$.

The idea of this theorem is that the two groups Γ_1 and Γ_2 "decouple" (Patterson's own wording). The groups Γ_1 and $h_i \Gamma_2 h_i^{-1}$ will have less interaction with each other as h_i pulls Γ_2 away from Γ_1. As a consequence the free product of these Kleinian groups results in another Kleinian group with a limit set of Hausdorff dimension arbitrarily close to that of one of the groups making up the free product. This yields the following corollary.

Corollary 1.9. *Suppose* Γ *is a non-elementary Kleinian group. For each* $\varepsilon > 0$ *there exists a rank two Schottky subgroup* Γ_ε *of* Γ *so that* $\dim(\Lambda(\Gamma_\varepsilon)) < \varepsilon$.

Here, we will work out fine details of how one can achieve this corollary. Patterson's theorem will be used implicitly during the process when we discuss the geometry and calculus that is used. Such mathematics is worthwhile as constructions of Kleinian groups with an analysis of their critical exponents and Hausdorff dimension of their limit sets have appeared (see [2], [3], [4], [11], [12], and [13] for some examples). In fact, the article [11] (and specifically Theorem 1.8) responds to [3]. In [3], critical exponents of free products of Kleinian groups were compared to the critical exponents of the groups used in the free product. The comparison used explicit calculations of critical exponents which differs from the use of Patterson's ideas using a more geometric approach. We hope that this method of construction can be used in more constructions of Kleinian groups where the Hausdorff dimension of the limit set is of interest and perhaps even generalized to other spaces as well, see [14].

Note that we proceed with proof for Kleinian groups acting on $\mathbb{B}^3$. We feel that this is appropriate for multiple reasons. First, it was shown by Hou in [4] that Kleinian groups acting on $\mathbb{B}^3$ with limit sets of small Hausdorff dimension are classical Schottky groups. In addition, the major theorems that we use here were originally done in this dimension, i.e. Theorem 1.5 and Theorem 1.7; we wish to work in this traditional setting. Finally, this allows for ease of exposition and highlights low dimensional geometry.

In the case of Kleinian groups of higher dimensions, examples of subgroups with limit sets of small Hausdorff dimension exist as well. We refer the reader to [13] for one such construction. There, infinitely generated Schottky groups of the second kind are created to obtain the desired properties.

2. Construction

Finally, we now provide a detailed proof of Corollary 1.9.

Proof of Corollary 1.9. Let Γ be a non-elementary Kleinian group. It contains a rank two Schottky subgroup $\widetilde{\Gamma} = \langle \gamma_1, \gamma_2 \rangle$. Such a group can be constructed in the following way. Consider isometric circles of two loxodromic elements in Γ and take high enough powers of the elements, if necessary, so that none of the isometric circles intersect. One may then apply Theorem 1.3 to the two cyclic groups generated by the loxodromic elements.

If necessary conjugate so that zero is in $CH(\Lambda(\widetilde{\Gamma}))$. Let $M = \mathbb{B}^3/\widetilde{\Gamma}$ and $\pi : \mathbb{B}^3 \to M$ be the canonical map. Let D be a subset of $C(\widetilde{\Gamma})$ so that it contains $\pi(0) = x \in M$ and $C(\widetilde{\Gamma}) - D$ is separated into disjoint sets, where one set contains the closed geodesic corresponding to γ_1 and the other contains the closed geodesic corresponding to γ_2. Since $\widetilde{\Gamma}$ is a Schottky group, $C(\widetilde{\Gamma})$ is compact and hence D has finite diameter, call it L. See Fig. 1.

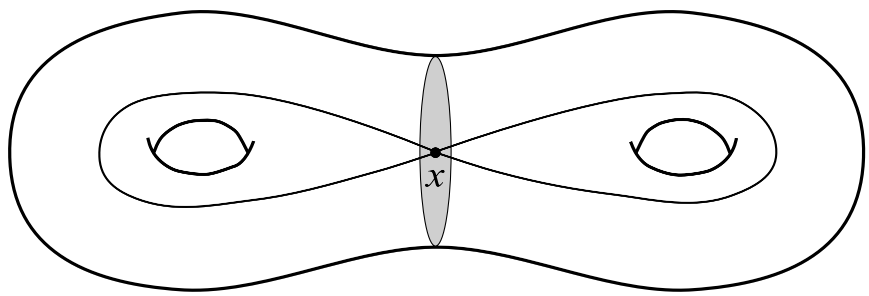

Fig. 1: The region D is shaded. The two loops represent homotopically non-trivial paths of γ_1 and γ_2 which begin and end in $x \in D$.

Let β_i be the closed geodesic associated to γ_i. Assume that $l(\beta_1) \leq l(\beta_2)$ where $l(\cdot)$ denotes the length of the geodesic. Then β_1 is the systole (the shortest closed geodesic) of M. Note that this is possible because of the way we constructed $\widetilde{\Gamma}$. Given any Schottky group the generators may not provide a systole. However, in constructing $\widetilde{\Gamma}$ by using isometric circles, the systole corresponds to one of the generators.

If w_m is a word of length m in $\widetilde{\Gamma}$ we would like to show that $d(0, w_m(0)) = l(\pi[0, w_m(0)]) \geq ml(\beta_1) - mL$. A geodesic segment that connects 0 and $w_m(0)$ in $\mathbb{B}^3$ corresponds to a loop β in M that begins and ends at x. A word of length m produces β so that it may pass through D at most m times. Now we will divide β into a countable number of sub-paths. Let p'_j denote a path that begins at one of the points $\beta \cap D$, traverses in M so that it is homotopically non-trivial, and ends at the next occurrence of $\beta \cap D$. For a word of length m, there are at most m many p'_j. Now let p_j be the simple closed curve that is the union of p'_j and the line segment connecting the endpoints of p'_j. Clearly $l(p_j) \geq l(\beta_1)$ for each j as β_1 is the systole. Then $l(\beta) + mL \geq l(p_1) + \cdots + l(p_m) \geq ml(\beta_1)$. It is possible that there may be $k < m$ many p_j and in this case there must be at least one p_j where the curve loops around a genus more than once. In other words, the loop is homotopic to q cycles of β_i where q is an integer greater than 1. Then $l(p_j) \geq ql(\beta_i)$. Let q_j be the cycles of β_i that correspond to p_j. Then we still obtain the same result: $l(\beta) + mL \geq l(p_1) + \cdots + l(p_k) \geq q_1 l(\beta_1) + \cdots + q_k l(\beta_1) \geq ml(\beta_1)$ where $q_1 + \cdots + q_k = m$. Therefore we have $d(0, w_m(0)) = l(\pi[0, w_m(0)]) = l(\beta) \geq ml(\beta_1) - mL$.

Now consider the Poincaré series for $\widetilde{\Gamma}$,

$$P_s = \sum_{\gamma \in \widetilde{\Gamma}} e^{-sd(0,\gamma(0))}.$$

If $s > \delta(\widetilde{\Gamma})$ then the series converges and we can rearrange the terms. We rewrite P_s as:

$$1 + \sum_{m=1}^{\infty} \left(\sum_{w_m \in \widetilde{\Gamma}} e^{-sd(0,w_m(0))} \right)$$

which is the sum of all length 1 words plus the sum of length 2 words, etc. From the remarks above, for a fixed m we have $e^{-sd(0,w_m(0))} \leq e^{-sm(l(\beta_1)-L)}$. Since $\widetilde{\Gamma}$ is generated by two elements we have a choice of $(4)(3)^{m-1}$ length m words. Thus the sum above is less than or equal to:

$$1 + \sum_{m=1}^{\infty} (4 \cdot 3^{m-1}) e^{-sm(l(\beta_1)-L)} = 1 + \frac{4}{3} \sum_{m=1}^{\infty} 3^m e^{-sm(l(\beta_1)-L)}.$$

The sum is a geometric series where $a = b = \frac{3}{e^{s(l(\beta_1)-L)}}$ and the series converges to $\frac{a}{1-b}$ if $b < 1$ or diverges if $b \geq 1$.

For any positive real number k, by taking subgroups of $\widetilde{\Gamma}$, say $\widetilde{\Gamma}_j = \left\langle \gamma_1^j, \gamma_2^j \right\rangle$ there is a j large enough, so that the minimal geodesic in $\widetilde{\Gamma}_j$, $\widetilde{\beta}_1$, has length greater than k. Note that $\widetilde{\Gamma}_j$ for any j is also a Schottky group as $\widetilde{\Gamma}_j$ is finitely generated, purely loxodromic, and $\Omega(\widetilde{\Gamma}) \neq \emptyset$ since it is a subset of $\widetilde{\Gamma}$. Furthermore, $\left\langle \gamma_i^j \right\rangle$ is a subset of $\langle \gamma_i \rangle$ for $i = 1, 2$ and thus, we have satisfied the conditions of Corollary 4.1.2 in [6] and $\widetilde{\Gamma}_j$ is free. Then by Theorem 1.2, $\widetilde{\Gamma}_j$ is also a Schottky group.

Now let $s = \varepsilon > 0$. One may find a k large enough so that $\frac{3}{e^{s(k-L)}} < 1$. Specifically we want $k > \frac{\ln(3)}{s} + L$. By picking j large enough, the subgroup

$\widetilde{\Gamma}_j = \Gamma_\varepsilon$ will have the minimal geodesic $\widetilde{\beta}_1$ greater than k so that $b < 1$. It follows that $\delta(\Gamma_\varepsilon) < \varepsilon$.

From Proposition 1.1 we know that Γ_ε is geometrically finite. Therefore, by Theorem 1.5, $\dim(\Lambda(\Gamma_\varepsilon)) = \delta(\Gamma_\varepsilon)$ and hence $\dim(\Lambda(\Gamma_\varepsilon)) < \varepsilon$. $\square$

Corollary 2.1. *Suppose Γ is a non-elementary Kleinian group. For each $\varepsilon > 0$ and each $r \geq 2$ there exists a rank r Schottky subgroup Γ_ε of Γ so that $\dim(\Lambda(\Gamma_\varepsilon)) < \varepsilon$.*

Proof. By Corollary 1.9 there exists a rank two Schottky subgroup $\widetilde{\Gamma}$ of Γ so that the Hausdorff dimension of the limit set is less than ε. Since rank two Schottky groups are non-elementary, they contain Schottky subgroups of rank r for any $r \geq 2$. Let Γ_ε be a rank r Schottky subgroup of $\widetilde{\Gamma}$. Since $\widetilde{\Gamma}$ has Hausdorff dimension of the limit set less than ε and Γ_ε is a subgroup of this group, $\dim(\Lambda(\Gamma_\varepsilon)) < \varepsilon$. $\square$

Corollary 2.2. *Suppose Γ is a non-elementary Kleinian group. For each $\varepsilon > 0$ there exists an infinitely generated free subgroup Γ_ε of Γ so that $\dim(\Lambda(\Gamma_\varepsilon)) < \varepsilon$.*

Proof. From Corollary 1.9 there exists a Schottky subgroup $\widetilde{\Gamma}$ of Γ with $\dim(\widetilde{\Gamma}) < \varepsilon$. Furthermore, a finitely generated free group contains infinitely generated free subgroups (see [6]). Therefore, $\dim(\Gamma_\varepsilon) < \varepsilon$. $\square$

3. Acknowledgements

This work constitutes part of a doctoral thesis written at Wesleyan University. We thank all the professors at Wesleyan for their support. In addition, we thank the referee for helpful comments that nicely contributed to this paper's exposition.

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