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Exploring Hausdorff Dimension

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Abstract

We live in a 3-dimensional world. We know 2-dimensional objects as those we can see on paper. But what would an object with a nonnatural number dimension look like? These dimensions exist and some unique sets have such dimensions, like $\frac{log2}{log3}$. We will examine Hausdorff dimension, the Cantor Set, and multiple manipulations of it.

1 Introduction

In this paper, we will primarily look at the middle third Cantor set. This set has dimension $\frac{log2}{log3}$, using calculations of Hausdorff measure and Hausdorff dimension. We fully prove the middle third Cantor set has such dimension in a thorough proof. My research consists of changes within the Cantor set and how those changes affect dimension. We will change: its first interval's length, its removal size at each interval, and its shape. All of these factors differently affect the set's dimension. Lastly, we try to visualize how Hausdorff dimension would hold in hyperbolic space.

2 Background

Hausdorff dimension is a concept from topology. Theorized by Felix Hausdorff in 1918, "the Hausdorff dimension is a counting number agreeing with a dimension corresponding to its topology." In order to fully understand this, we must first discuss our basic knowledge of dimension. We learned in grade school the dimension of a line is 1, a flat square 2, and a cube 3. Hausdorff argued the dimension of a single point is zero. He wanted to further examine complex sets in order to refer to its size. Such "complex sets" are usually irregular and extended from simple shapes or intervals in $\mathbb{R}^n$.

3 Basics

In order to discuss Hausdorff Dimension, we first need to look at measures.

Definition 3.1. A *measure* m is a nonnegative real function from a set $F \to \mathbb{R}$

such that $m(\emptyset) = 0$ where $\emptyset$ is the empty set, and

$$m(A) = \sum_n m(A_n)$$

for any infinite or countable collection of pairwise disjoint sets (A_n) in F such that $A = \bigcup(A_n)$ is also in F.

In other words, a measure gives a number to a set in order to reference its size. Hausdorff measure is more complicated and therefore requires more machinery.

Definition 3.2. Let U be a non-empty subset of n-dimensional Euclidean space. Then the diameter, $|U|$ is equal to

$$|U| = sup\{|x - y| : x, y \in U\}.$$

Definition 3.3. A *cover* is any family of subsets of a given set B whose union is B.

A cover is different from a measure, though the two are similar. A cover does not assign a number – rather, it assigns a set. This is how you can physically "cover" a set where the union is the whole space. For example, the set [0,1] has multiple covers, including:

a) $[0, \frac{1}{n}]$ where $n \in \mathbb{N}$

b) $[0, \frac{1}{2}] \cup (\frac{1}{2}, 1]$

and so on.

Definition 3.4. If $\{U_i\}$ is a countable or finite collection of sets of diameter at most δ that cover F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \leq |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a *δ-cover* of F.

In other words, imagine covering a set of small dots with circles of diameter at most δ (see Figure 1). We enhance the image of F to see two possible δ-covers of F. Let's look at the cover on the lower left of Figure 1. Clearly, the union of all these smaller circles is equal to F. Because $0 \leq |U_i| \leq \delta$, then this set with diameters $\{U_i\}$ is a possible δ-cover of F. We use the word "possible" because there are many other ways to cover F with other sizes of δ.

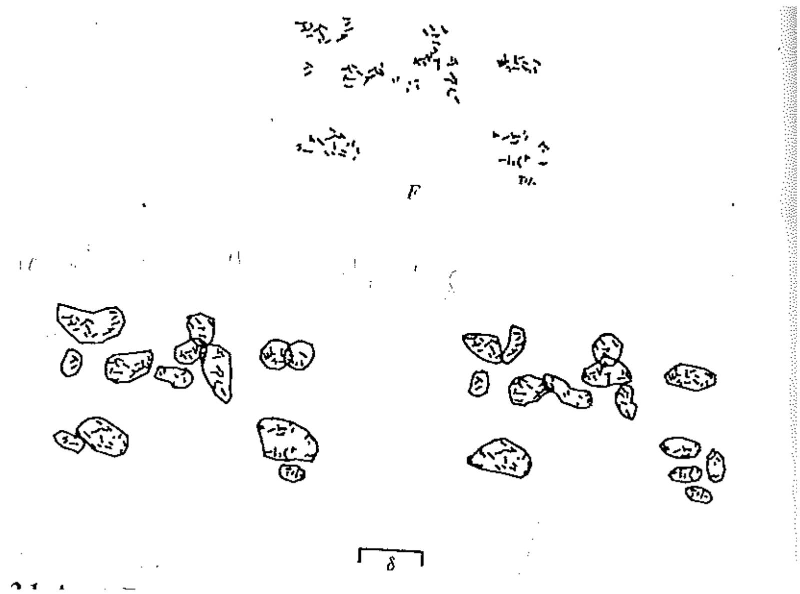

Figure 1: A set F with two possible δ-covers. This figure is taken from [1].

Now that we have the necessary tools, we can begin discussing Hausdorff measure.

Definition 3.5. *Hausdorff Measure*. If F exists in $\mathbb{R}^n$ and $s > 0$, then for any $\delta > 0$,

$$\mathcal{H}^s_\delta(F) = inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta \text{ -cover of } F \right\}.$$

Essentially, the Hausdorff measure is the smallest number of the sums of the smallest diameters less than δ capable of covering F.

Therefore, considering each δ-cover of F, we try to minimize the summation of the diameters by minimizing the size of s. Then as s gets smaller, so does the Hausdorff measure of F. But when δ becomes smaller, the amount of covers of F also decreases. This is intuitive. However, the infimum of $\mathcal{H}^s(F)$ is non-decreasing. This isn't as intuitive. Here is a clarification. Consider the set $\mathbb{N}$. The infimum of this set is 1. Now, take away one number in $\mathbb{N}$ but don't say which. What can we say about the infimum? It is either the same or higher, hence the infimum is non-decreasing. The same thing is happening with the amount of valid covers of F as δ decreases.

Going back to the measure, as $\delta \to 0$ and the infimum of $\mathcal{H}^s_\delta(F)$ increases (is non-decreasing), then

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F). \tag{1}$$

This equality is called *the s-dimensional Hausdorff measure of F*. We find the limit in order to extract δ to get a general Hausdorff measure to help us find s. Any set F (where $F \subset \mathbb{R}^n$) has a Hausdorff measure and its limit exists, though it is usually zero or ∞.

Hausdorff measure is difficult to calculate. However, we can get close to a universal equation for whole number, low-dimensional subsets of $\mathbb{R}^n$.

1. The number of points in a set F is given by $\mathcal{H}^0(F)$, where $dim_H F = s = 0$ and F is countable.

2. The length of a smooth curve F is given by $\mathcal{H}^1(F)$, where $dim_H F = s = 1$.

3. A smooth surface F is given by $\mathcal{H}^2(F) = (4/\pi) \times area(F)$, where $dim_H F = s = 2$.

4. A shape F is given by $\mathcal{H}^3(F) = (6/\pi) \times vol(F)$, where $dim_H F = s = 3$.

Hausdorff Dimension

The s-dimensional Hausdorff measure exists and we are able to find the value of s for any set $F \subset \mathbb{R}^n$. We can take this a step further to show if $t > s$ and $\{U_i\}$ is a δ-cover of F, then

$$\sum_i |U_i|^t \leq \sum_i |U_i|^{t-s} |U_i|^s.$$

By the triangle inequality, the above equation works. We can go one step further to say

$$\sum_i |U_i|^{t-s} |U_i|^s \leq \delta^{t-s} \sum_i |U_i|^s.$$

Since $\{U_i\}$ is a δ-cover of F, we know $|U_i| \leq \delta$ by definition 3.4.

Using definition 3.5, the infima show

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F).$$

As $\delta \to 0$ and if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$ when $t > s$. This follows from Definition (3.5), equation (1), and substitution.

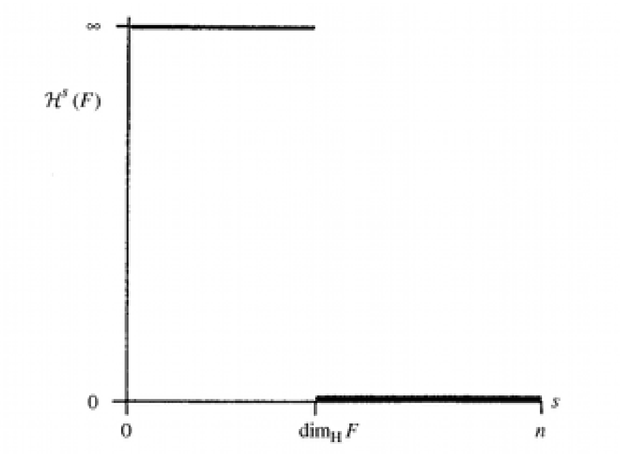

Figure 2: A graph of $\mathcal{H}^s(F)$ against s.

We can create a graph of $\mathcal{H}^s(F)$ against s (Figure 2) to see there exists a critical value of s where $\mathcal{H}^s(F)$ is neither 0 nor ∞. In Figure 2, this is the value $dim_H F$. There is only one value of s for each and any unique set F. We call this *the Hausdorff dimension of* F, written $\dim_H F$. Recall Definition (3.5) and see the value $s = dim_H F$. Therefore $\mathcal{H}^s(F)$ will always satisfy $0 \leq \mathcal{H}^s(F) \leq \infty$.

Let's look at a calculation of Hausdorff dimension.

Example. Let F be a flat disk with unit radius in $\mathbb{R}^3$. We know the dimension of

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this disk is greater than zero since it's more than one single point. We also know the length of a disk is infinite because it doesn't have endpoints. Therefore:

$$\mathcal{H}^1(F) = length(F) = \infty$$

$$0 < \mathcal{H}^2(F) = (4/\pi) \times area(F) = 4 < \infty$$

$$\mathcal{H}^3(F) = (6/\pi) \times vol(F) = 0$$

Therefore $dim_H F = 2$ and $\mathcal{H}^s(F) = \infty$ when $s < 2$ and $\mathcal{H}^s(F) = 0$ when $s > 2$.

Later, we will see how the length, area, and volume of a set may not always be consistent or easily determined. This yields a dimension between two whole numbers.

As with all new mathematical terms, Hausdorff dimension holds similar properties to that of most dimension definitions. They are:

Scaling Property: Let S be a similarity transformation of scale factor $\lambda > 0$. If $F \subset \mathbb{R}^n$, then $\mathcal{H}^s(S(F)) = \lambda^s \mathcal{H}^s(F)$.

Monotonicity: If $E \subset F$ then $dim_H E \leq dim_H F$, which directly comes from equations above.

Countable Stability: If $F_1, F_2, ...$ is a countable sequence of sets then $dim_H \bigcup_{i=1}^{\infty} F_i = sup_{1 \leq i < \infty}\{dim_H F_i\}$. Essentially, dimension of the infinite union of F_i is equal to the dimension supremum F_i

4 Hausdorff Dimensions of Sets

One of the most famous examples using Hausdorff dimension is the middle third Cantor set. Picture the segment $[0, 1]$. Call this line E_0. For E_1, right below E_0, remove the middle third of E_0, such that E_1 is the segments $[0, \frac{1}{3}]$, $[\frac{2}{3}, 1]$. This repeats to infinity.

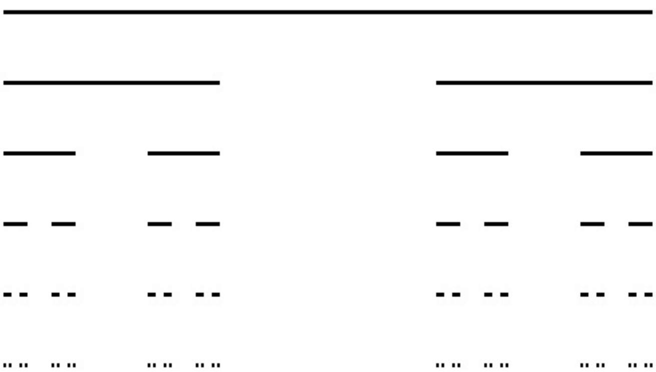

Figure 3: The middle third Cantor set.

The middle third Cantor set F is composed of the numbers in E_n for all $n \in \mathbb{N}$. Formally, F is the intersection $\bigcap_{n=0}^{\infty} E_n$. F is so infinitely small that it would be hard to physically draw the set. Therefore, we can consider the middle third Cantor set F to be $\lim_{n\to\infty} E_n$ and pictures of F to be pictures of some E_n.

What can we say about the dimension of the Cantor set? F is clearly more than one point by construction. However, the length of F is infinite because F as a set is infinite (though restricted between $[0,1]$). Furthermore, every subinterval E_n is a subset of a one-dimensional line $[0,1]$. Hence, we know $0 < s < 1$. This is a clear sign we need to use Hausdorff dimension!

Theorem 4.1. *The Hausdorff dimension of the Cantor Set is* $\frac{log2}{log3} = 0.6309...$

An intuitive proof would go as follows.

Proof. The Cantor Set, call it F, becomes two parts: a left side $F_L = F \cap [0, \frac{1}{3}]$ and a right side $F_R = F \cap [\frac{2}{3}, 1]$. We know both F_L and F_R are the same length and are a ratio to F by $\left(\frac{1}{3}\right)$. They are geometrically similar so $F = F_L \cup F_R$. Therefore, for any value s we can see:

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(F) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(F). \tag{2}$$

Because we are looking for a critical value $0 < s < \infty$ we are able to divide by $\mathcal{H}^s(F)$ to see $1 = 2\left(\frac{1}{3}\right)^s$. Applying log properties we get $s = \frac{log2}{log3}$. □

This explanation yields the Hausdorff dimension, but we need a rigorous proof in order to prove the Hausdorff measure supports the dimension we found.

Proof. Let E_n be the intervals that make up the Cantor set F. We call E_n *level-n intervals*. Then for any $n \geq 0$ we have E_n consisting of 2^n level-n intervals, each with length 3^{-n}. Take a 3^{-n}-cover over the intervals of E_n. We see:

$$\mathcal{H}^s_{3^{-n}}(F) \leq 2^n 3^{-ns}. \tag{3}$$

Letting $s = \frac{log2}{log3}$ in equation (3), we find $\mathcal{H}^s_{3^{-n}}(F) \leq 1$. As $n \to \infty$ then $\mathcal{H}^s(F) \leq 1$.

Now we prove $\mathcal{H}^s(F) \geq \frac{1}{2}$. We want to ultimately end up with the equation:

$$\sum |U_i|^s \geq \frac{1}{2} = 3^{-s} \tag{4}$$

for any cover $\{U_i\}$ of F. This equation will reduce to $\mathcal{H}^s(F) \geq \frac{1}{2}$ when $s = \frac{log2}{log3}$. That will allow us to always have an infimum greater than $\frac{1}{2}$ and therefore always yields a Hausdorff measure greater than $\frac{1}{2}$.

For each $\{U_i\}$, let n be an integer so that:

$$3^{(-n+1)} \leq |U_i| < 3^{-n}. \tag{5}$$

Equation (5) just uses definition 3.4 to ensure each cover is less than the next. From equation (5), we see U_i can intersect at only one level-n integer because each level-n integer is separated by a distance of at least 3^{-n}.

In equation (3), we saw $2^n 3^{-ns} = 1$ when $\frac{log2}{log3}$. Then $2^{-n} = 3^{-ns}$.

Let $j \geq n$. Then by construction $\{U_i\}$ intersects at most

$$2^{j-n} = 2^j 2^{-n} = 2^j 3^{-sn} \leq 2^j 3^s |U_i|^s \tag{6}$$

level-j intervals of E_j. Equation (6) comes from what we know about inequalities and that $2^n 3^{-ns} = 1$ when $s = \frac{log2}{log3}$. The right side of the equation, which we have not yet considered, comes from equation (5). We know $|U_i| < 3^{-n}$, so when we raise these to the sth power, we have the opposite.

Continuing on with the proof, we can find a j big enough to satisfy $3^{(-j+1)} \leq |U_i|$ for all $\{U_i\}$, just as we did in equation (4). Then the selected $\{U_i\}$ intersect each 2^j basic intervals with length 3^{-j}. When we count these intervals, we get:

$$2^j \leq \sum_i 2^j 3^s |U_i|^s. \tag{7}$$

Divide by 2^j and divide again by 3^{-s} to see equation (5) is equal to equation (2).

□

It may seem like we only proved something about the Hausdorff measure, that it is somewhere between or equal to $\frac{1}{2}$ and 1. Remember that this works because we found an s to satisfy $\frac{1}{2} \leq \mathcal{H}^s(F) \leq 1$, which is also true about $dim_H F$. From the definition of Hausdorff dimension, we saw that we need to satisfy $0 \leq \mathcal{H}^s(F) \leq \infty$. The critical value of s where $\mathcal{H}^s(F)$ is not 0 or ∞ is how we find the Hausdorff dimension of F, just as we have proven above.

5 Manipulations of Hausdorff Dimension Sets

Length

What happens if we examine the Cantor set and change its size? Let F be the middle third Cantor set where $E_0 = [0, 3]$. We will still remove the middle third, such that E_1 is the set $[0, 1] \cup [2, 3]$. Therefore, we still have a scaling factor of $\left(\frac{1}{3}\right)$ from E_k to E_{k-1}. We set up an intuitive calculation:

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(F) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(F).$$

Notice this is exactly equation (2). Hence, we still have $s = \frac{log2}{log3}$ and therefore the Hausdorff dimension stays the same despite our change of endpoint from 1 to 3.

Now let $X = [0, \pi]$. Remove the middle third once again, and continue removing the middle third of each line from E_k to E_{k+1}. Then $E_1 = [0, \frac{\pi}{3}] \cup [\frac{2\pi}{3}, \pi]$. We have the exact same idea with this set as the Cantor set, just with length π instead of length 3. Again, we have a scaling factor of $\left(\frac{1}{3}\right)$ from E_k to E_{k+1} and note we again have equation (2), based on the math in the previous paragraph.

Therefore, Hausdorff dimension of the Cantor set does not change based on length of the first interval (in these cases, the length of E_0). This may seem hard to believe, so let me explain it in a different, less-mathematical way. Choose any natural number, say 42. Let's take a look at E_{42} in the normal middle third Cantor set. We can find the length of E_{42} by using 3^{-n}, so the length is $\left(\frac{1}{3^{42}}\right)$. Furthermore, there are 2^{42} intervals at this level of the Cantor set. Imagine that we relabel E_{42} as the beginning of an entirely new middle third Cantor set A. Then $E_{42} = A_0$.

Though we have a separate set A, the dimension of A does not change based on the length of A_0 or E_{42}. We still take away the same scaled amount $\left(\frac{1}{3}\right)$ from A_n and have the same amount of intervals (2^n). Hence, the dimension of A is still $\frac{log2}{log3}$. Furthermore, nothing limits the middle third Cantor set from being part of some bigger set B, just looking at the $[0, 1]$ portion.

Hyperbolic Lengths

Hyperbolic geometry is a type of geometry where Euclid's parallel postulate does not hold. It is changed to the following: *For any given line R and point P not on R, in the plane containing both line R and point P there are at least two distinct lines through P that do not intersect R.* Lines and shapes look different in hyperbolic space. Lines are more circular typically.

Remark. The Hausdorff dimension of any set G in Euclidean geometry has the same Hausdorff dimension in hyperbolic geometry.

Consider a circle with $r = 1$. Then the circumference $C = 2\pi r = 2\pi$. Look at the top half of this circle, and create the middle third Cantor set from it so $G_0 = [0, \pi]$. Hence, $G_1 = [0, \frac{\pi}{3}] \cup [\frac{2\pi}{3}, \pi]$. To find the Hausdorff dimension of G, note the scaling factor of $\left(\frac{1}{3}\right)$ from G_k to G_{k+1}. Then

$$\mathcal{H}^s(G) = \mathcal{H}^s(G_L) + \mathcal{H}^s(G_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(G) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(G).$$

This is similar to the last section. After dividing by $\mathcal{H}^s(G)$, we again end up with $1 = 2\left(\frac{1}{3}\right)^s$ to see $s = \frac{log2}{log3}$.

Though we need two dimensions to see a half-circle, the removal size and lengths are the same between G and X. But we can fit G in the first dimension if we flatten it so that G becomes X. Therefore, the dimensions are the same, where $s = \frac{log2}{log3}$. If we looked at G as a full circle instead of a half-circle, then the dimension would be different because we would always need at least two dimensions to see G. Therefore, s would probably be greater than 1.

Removal Size

How does removing the middle third differ from removing the middle ninth in the Cantor set? Let Y be the line $[0,1]$. Remove the middle ninth from Y and place the remaining two intervals directly below Y such that $Y_1 = [0, \frac{4}{9}] \bigcup [\frac{5}{9}, 1]$. Repeat this process for Y_n to infinity.

For a quick intuitive proof, let's again break apart Y_L and Y_R such that $Y_L = Y \cap [0, \frac{4}{9}]$ and $Y_R = Y \cap [\frac{5}{9}, 1]$. Each new interval is scaled to Y by a ratio of $\frac{4}{9}$. Therefore, we can use the same set-up for the equation:

$$\mathcal{H}^s(Y) = \mathcal{H}^s(Y_L) + \mathcal{H}^s(Y_R) = \left(\frac{4}{9}\right)^s \mathcal{H}^s(Y) + \left(\frac{4}{9}\right)^s \mathcal{H}^s(Y).$$

Divide by $\mathcal{H}^s(Y)$ and we have $1 = 2\left(\frac{4}{9}\right)^s$. Then $s = \frac{-log2}{2(log2 - log3)}$.

We can assume that whenever we change the removal size of the middle piece in the Cantor set, then the dimension of the Cantor set changes. This works out because the scaling ratio is different, hence creating a large difference in the equation to find s.

Further Research

The middle third Cantor set can be manipulated in many different ways. For example, what happens if we let $k \in \mathbb{N}$ and remove k segments from the interval $[0, 1]$, leaving a space between the segments of size 3^{-k}? Would the set have no dimension, for each interval with removal size 3^{-k} would have a different dimension? I also looked at the plane analogue Cantor Dust set, which happened to have dimension 1. With this Cantor Dust, the image remains the same size but smaller pieces get added with every new subset, whereas the middle third Cantor set removes pieces with every new subset. The possibilities are endless with Hausdorff dimension, for we have no solid foundation for how dimension is changed based on small changes of a set.

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