



Sacred Heart
UNIVERSITY

Sacred Heart University
DigitalCommons@SHU

Academic Festival

Apr 21st, 12:30 PM - 1:45 PM

An Exploration of the Minimum Clue Sudoku Problem

Lauren Puskar
puskarl@mail.sacredheart.edu

Follow this and additional works at: <http://digitalcommons.sacredheart.edu/acadfest>

Puskar, Lauren, "An Exploration of the Minimum Clue Sudoku Problem" (2017). *Academic Festival*. 20.
<http://digitalcommons.sacredheart.edu/acadfest/2017/all/20>

This Essay is brought to you for free and open access by DigitalCommons@SHU. It has been accepted for inclusion in Academic Festival by an authorized administrator of DigitalCommons@SHU. For more information, please contact ferribyp@sacredheart.edu.

Puskar: An Exploration of the Minimum Clue Sudoku Problem

SACRED HEART UNIVERSITY

MA-398-A

SENIOR SEMINAR IN MATHEMATICS

An Exploration of the Minimum Clue Sudoku Problem

Author:

Lauren PUSKAR

Supervisor:

Dr. Bernadette BOYLE

December 12, 2016

1 Abstract

This paper explores the "Minimum Sudoku Problem," that says there must be at least 17 clues in order for a Sudoku Board to have a unique solution. We prove uniqueness up to seven clues for 9x9 boards. We also take a look at the different patterns of 4x4 boards, and how graph theory and the coloring of a graph relates to solving a Sudoku puzzle.

2 Introduction

When we solve a common Sudoku puzzle, we notice that there are at least 17 clues given. However, what would happen if we were given less than 17 clues? Would we be able to come up with a unique and valid board? The answer to this question is NO! If we are given less than 17 clues, it is impossible to come up with a unique and valid board. Gary McGuire, Bastian Tugemann and Gilles Civarioz have already exhaustively proven this using a specific algorithm and computer program, which checked all the 6.7×10^{21} possible solutions of a Sudoku board [1]. This paper will take a look at some of their results in a simpler way by exploring the Sudoku boards that contain between 1,...,7 clues, and furthermore, proving why there does not exist a unique solution board for these such cases.

3 Background

3.1 Definitions

In order to fully understand the terms used in this paper, we will first start off with some definitions:

Definition 3.1. *A Sudoku puzzle is a 9x9 grid, some of whose cells already contain a digit between 1 and 9. The task is then to complete the grid by filling in the remaining cells such that each row, each column, and each 3x3 box within the bolded lines of the larger 9x9 box contains the digits from 1 to 9 exactly once.*

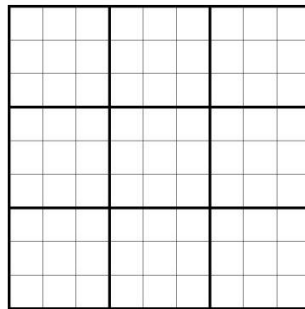


Figure 1: Example of an Empty Sudoku Board

Definition 3.2. *A complete board is a Sudoku board that satisfies all the rules of Sudoku.*

Definition 3.3. *A unique board is a Sudoku board that has only one completion.*

Definition 3.4. *A valid board is a Sudoku board for which there exists exactly one completion which follows all the rules of Sudoku.*

Note: Valid Boards can have empty cells.

Definition 3.5. A band is the set of rows 1-3, rows 4-6, or rows 7-9.

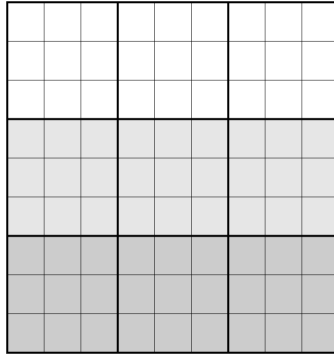


Figure 2: Example of a band

Definition 3.6. A stack is the set of columns 1-3, columns, 4-6, or columns 7-9.

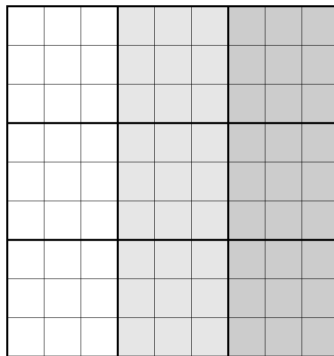


Figure 3: Example of a stack

Lemma 3.7. The pigeonhole principle states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.[1]

3.2 History

Gary McGuire of the University College Dublin was the first to show a legitimate proof of the minimum number of clues, or starting digits, needed to complete a Sudoku puzzle with a unique solution, which is 17. With the help of his team, including Bastian Tugemann and Gilles Civarioz, he was able to test all possible completed grids for every 16-clue puzzle. However, this took about seven million CPU hours and a full 365 days to complete at the Irish Centre for High-End Computing in Dublin. His computer used an algorithm that cut down the number of grids by looking for unavoidable sets, or arrangements of numbers within the completed puzzle that are interchangeable, and thus could result in multiple solutions. Then the clues must overlap, or "hit", the unavoidable sets which would prevent the unavoidable sets from causing multiple solutions. This then lead to a smaller computing task for his team to show that no 16-clue puzzle can hit them all. Currently, the only way to prove that no unique 16-clue puzzle exists is through this brute force way. However, many mathematicians are still working on different algorithms to prove such problem, including Gordon Royle of the University of western Australia [2].

4 9x9 Boards

After studying boards with exact numbers of starting clues, we were able to come up with the following theorems.

Theorem 4.1. *If we are given exactly one clue, called n , which is a number from the set $\{1, \dots, 9\}$, on a blank Sudoku board, then there is not a unique solution.*

Proof. Let n be a number from the set $\{1, \dots, 9\}$ that is placed in a random box on the Sudoku board. We can then create a complete board by filling in the rest of the boxes. Using proof by contradiction, assume that this complete board, G_1 , is unique. Now if we switch the two stacks that n is not in, we can create another completion, G_2 , that is different from G_1 . Therefore, G_1 is not unique. Thus, when we are given exactly one clue, our initial board is not valid. \square

Theorem 4.2. *If we are given up to exactly 5 clues, called n_1, n_2, n_3, n_4 and n_5 , which are numbers from the set $\{1, \dots, 9\}$, on a blank Sudoku board, then there is not a unique solution.*

Proof. Let n_1, n_2, n_3, n_4 and n_5 be five numbers from the set $\{1, \dots, 9\}$ that are placed in five separate boxes on the Sudoku board. This forms a completion that we will call G_1 . Since we have nine columns, at least four of those nine columns are guaranteed to be empty. Thus by the pigeonhole principle, after we place those four columns into three stacks we can see that one stack is guaranteed to have at

least two empty columns. Therefore, we can switch those two empty columns and create another completion called G_2 . Therefore, if we are given up to five clues the initial board is not valid. \square

As stated in the proof above, this theorem holds for boards that have exactly five clues or less. Thus, if we did not want to use Theorem 3.1, we can also use the one stated above to prove that no unique solution exists for a one-clue Sudoku board. This proof also applies to rows/bands as well.

Theorem 4.3. *If we are given up to exactly seven clues, called n_1, \dots, n_7 , which are numbers from the set $\{1, \dots, 9\}$, on a blank Sudoku board, then there is not a unique solution.*

Proof. Let n_1, \dots, n_7 be seven numbers from the set $\{1, \dots, 9\}$ that are placed in seven separate boxes on the Sudoku board. This forms a completion that we will call G_1 . Now since there are seven clues given, we know that there are at least two numbers which do not appear on the initial board and thus can be manipulated. Let us call these two numbers x and o . Now take the initial board and switch all the x 's to o 's and all the o 's to x 's to create another completion called G_2 . Therefore, if we are given up to seven clues the initial board is not valid. \square

This theorem can ultimately replace Theorem 3.1 and Theorem 3.2, as it is able to prove that no unique solution exists for up to seven clue-Sudoku boards. Unfortunately, I could not prove further than a seven-clue Sudoku board with

paper and pencil alone. However, I did change my course and took a look into 4x4 boards.

5 4x4 Boards

When we focused on a smaller board, specifically a 4x4 board, it was easier to see more patterns. First, we must understand the way in which we will be referring to them. When we talk about i , we are referring to the rows, and j refers to columns. So for example, if we say we have to switch $(1, 2)$ and $(3, 4)$ we are talking about the square that is of the first row, second column and the square of the third row, fourth column. The figure below highlights the squares I am referring to.

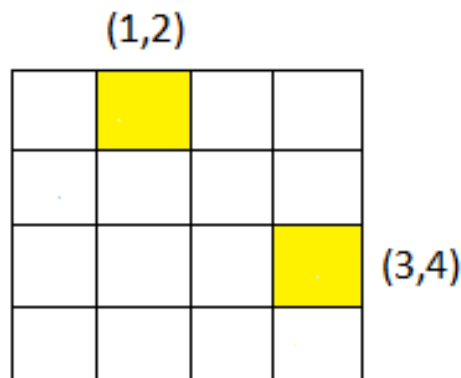


Figure 4:

Then when we talk about quadrants, the first quadrant is the upper right 2x2 square, the second is the upper left 2x2 square, the third is the lower left 2x2

square and the fourth one is the lower right 2x2 square. Figure 5 depicts it below.

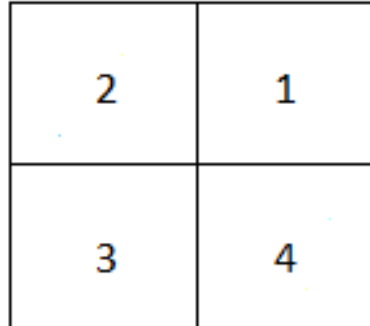


Figure 5: Quadrants

In studying various diagonal 4x4 boards, I came up with the following theorem.

Theorem 5.1. *If four clues are given along the diagonal of a 4x4 board and they need not be unique, then the solution board is not unique.*

Proof. Let the squares on the diagonals, $(1, 1)$, $(2, 2)$, $(3, 3)$ and $(4, 4)$, contain four numbers that need not be unique, but allow for at least one completion. That is some of the numbers can repeat more than once on the diagonal as long as they are not in the same quadrant. Then we can fill in the rest of the board properly according to the definition of Sudoku, which creates a completion called G_1 . Now we can form another completion that is different from G_1 , called G_2 , by using the same four clues on the diagonal. We can apply the definition of transpose, in which each column becomes its corresponding row and each row becomes its correspond-

ing column. We can do this because in the second quadrant of G_1 we see that $(1, 1)$ and $(2, 2)$ are filled with two different numbers, thus in order for it to be a Sudoku board it must not have any repeats within the quadrant. Furthermore, $(1, 2)$ and $(2, 1)$ must be different than $(1, 1)$ and $(2, 2)$ because we cannot have repetition either. However, since we want to form a board different than G_1 , $(1, 2)$ and $(2, 1)$ must contain different numbers than what they originally contained in G_1 . The only option we have is to switch them in order to not have repetition and to form a different board than G_1 . Then we must look at the rows in G_1 . Since the rules of Sudoku do not allow for repetition of numbers within rows we know that when we take the transpose of the first row of G_1 and make it the first column of G_2 there must also be no repetition. We continue this for each row and corresponding column. So we can take the transpose of the second row of G_1 and make it the second column of G_2 in which there must be no repetition because the second row of G_1 did not have repetition. Next we can take the transpose of the third row of G_1 and make it the third column of G_2 in which there must be no repetition because the third row of G_1 did not have repetition. Lastly, we can take the transpose of the fourth row of G_1 and make it the fourth column of G_2 in which there must be no repetition because the fourth row of G_1 did not have repetition. Thus, we see that this forms another completion that differs from our initial completion. Therefore, the initial board is not valid. \square

We notice that this works similarly if we took each column in G_1 and made it into a corresponding row in G_2 .

Below is an example of what this proof looks like for both unique and non-unique numbers on the diagonal.

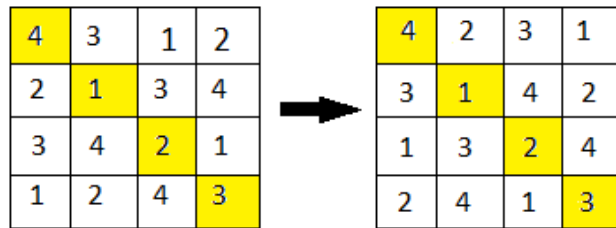


Figure 6: Unique Numbers

We also note, for unique numbers only, that G_2 can be formed by switching $(1, 2)$ and $(2, 1)$, $(3, 4)$ and $(4, 3)$, as well as rotating the numbers in the cells in quadrants one and three either clockwise or counterclockwise. We choose each quadrant's rotational direction by seeing if the rules of Sudoku are violated. For example, if there is repetition of a number in a row or column by rotating it clockwise, then we go the other direction counterclockwise.

4	3	2	1	→	4	1	2	3
1	2	3	4		3	2	1	4
2	1	4	3		2	3	4	1
3	4	1	2		1	4	3	2

Figure 7: Non-Unique Numbers

6 Graph Theory

Now let us look at a Sudoku board as a graph. When a board is complete we can translate it to a graph in which each vertex corresponds to a cell in the board and two distinct vertices are adjacent if and only if two cells share a row, column, or $n \times n$ block. Now some more definitions.

Definition 6.1. *A graph is a collection of points, called vertices, together with lines connecting (some of) them, called edges.*

Definition 6.2. *A proper graph coloring is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring.*

Definition 6.3. *A partially colored graph is the original puzzle with open squares, which means the graph representing it has yet-to-be-colored vertices.*

Definition 6.4. *A coloring that uses at most k colors is called k -coloring.*

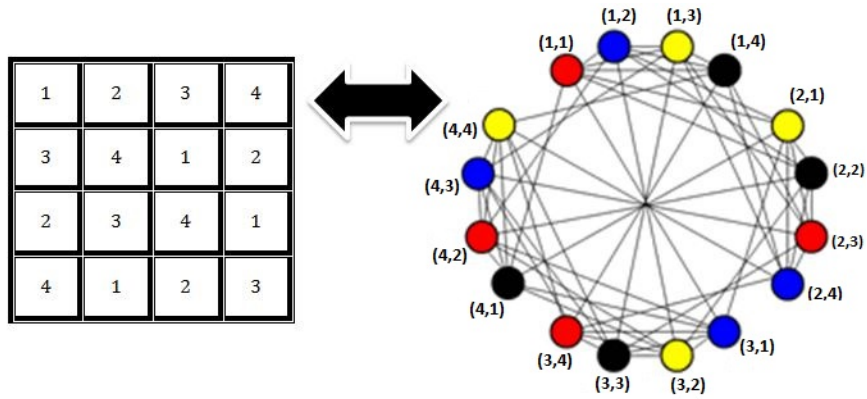


Figure 8: Graph Coloring

Definition 6.5. *An independent set to which a cell belongs is to be comprised of all the cells that include the same entry as the cell. [3]*

First we start of with an independent set of the puzzle below:

3	1		4
4		1	
1	3	4	2
2		3	1

Figure 9: Independent Set

$$"1" = [(3, 1)] = \{(3, 1), (2, 3), (1, 2), (4, 4)\}$$

$$”2” = [(4, 1)] = \{(4, 1), (3, 4)\}$$

$$”3” = [(1, 1)] = \{(1, 1), (3, 2), (4, 3)\}$$

$$”4” = [(2, 1)] = \{(2, 1), (1, 4), (3, 3)\}$$

The minimum number of colors required for a proper completion of any partial coloring is equal to the number of colors present in the partial coloring. For Sudoku this corresponds to the number of distinct digits appearing in the puzzle. Therefore, that would mean the puzzle in Figure 10, on the right hand side, needs at least four digits for a completion, which is the 4-coloring. The maximum number of colors which may appear in a proper coloring of a partial coloring is equal to the number of blank cells plus the number of colors already appearing in the graph. For Sudoku, this means that the puzzle in Figure 10, on the left hand side, needs a maximum of eight digits, which is the 8-coloring.[3]

Example below:

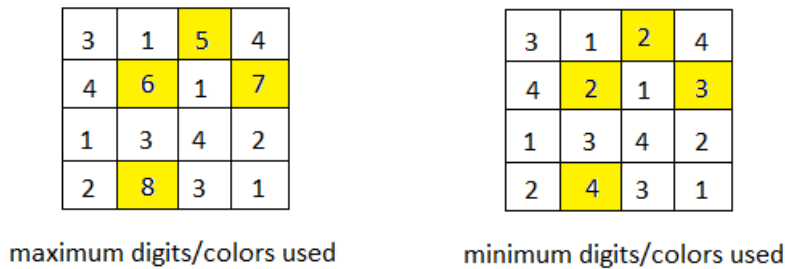


Figure 10: K-Colorings

We can form the empty cells into a graph in which the vertices are labeled with

their grid coordinates.

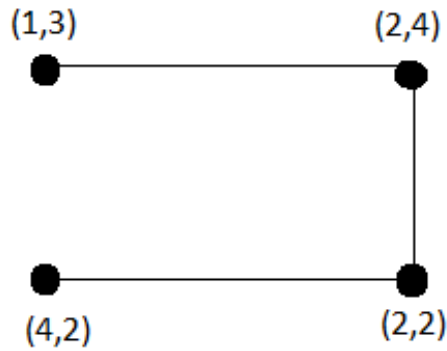


Figure 11:

Furthermore, we can create a partially colored graph, which will have vertices marked by the color classes that must be avoided.

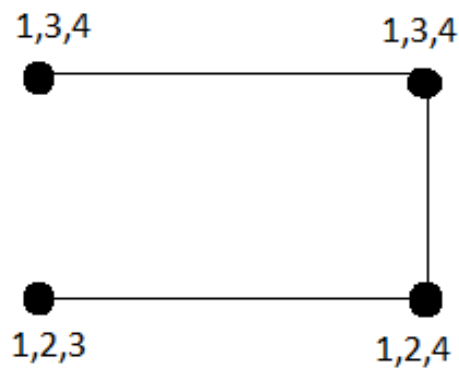


Figure 12: Partially Colored Graph

Once we have created our partially colored graph, we can then use the method of deletion and contraction to come up with the chromatic polynomial, $X(G, k)$,

which is the polynomial whose value at k is the number of proper colorings of a graph G using at most k colors. When applying deletion-contraction, any vertex formed by contracting an edge shares the adjacencies, and thus the coloring restrictions, of the formerly distinct vertices. Then the restricted color classes of a contracted-edge vertex correspond to the union of the restrictions of the distinct vertices. [3]

Now we will show how deletion-contraction works for our original board in Figure 9:

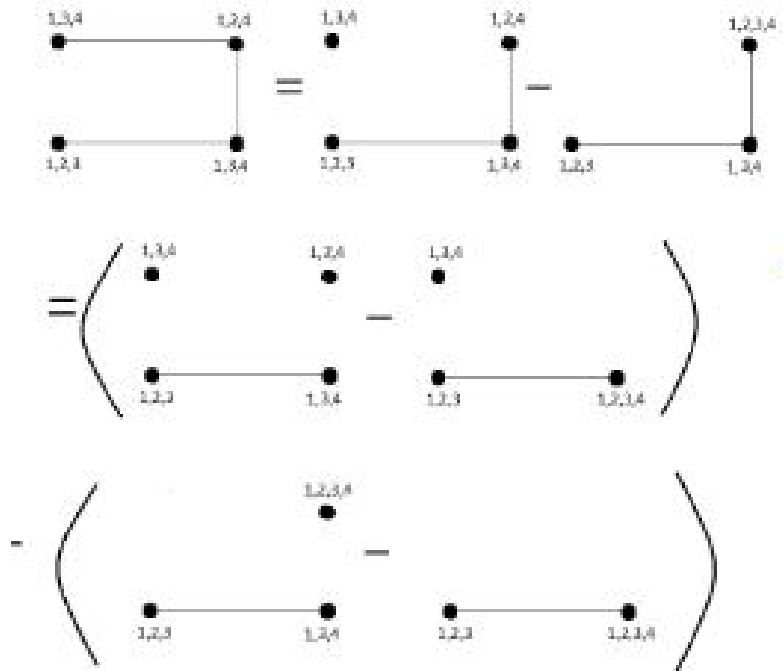


Figure 13: Deletion-Contraction Method

Depending on how many colors it has to avoid, we can find its chromatic

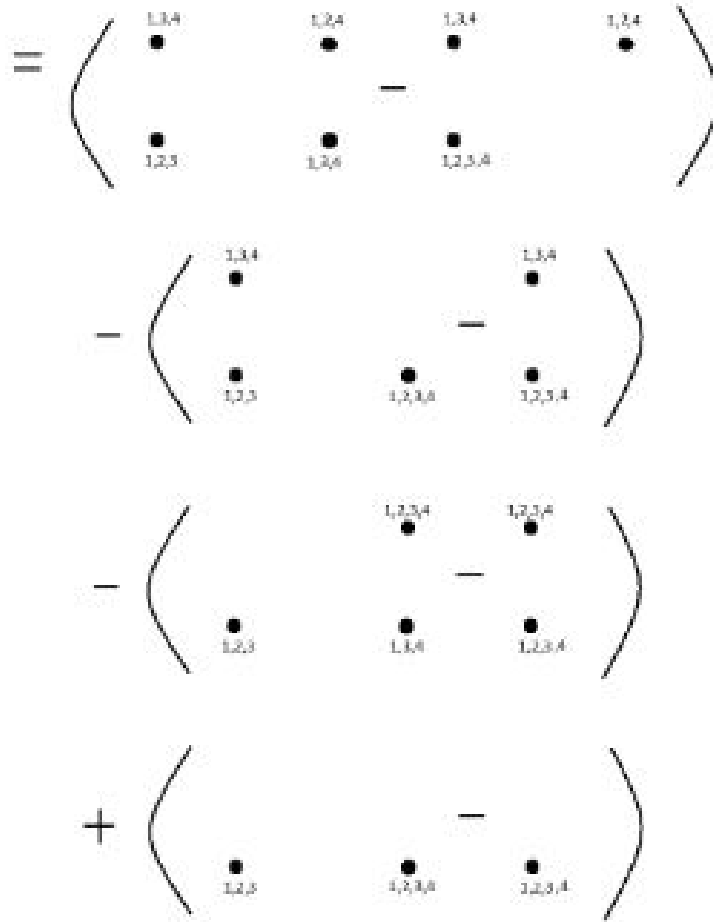


Figure 14: Deletion-Contraction Method Continued

polynomial. For our example above, we find that

$$X(G, k) = (k-3)^2(k-4)^2 - (k-3)^3(k-4)^2 - (k-4)^3 + (k-4)^2 + (k-3)(k-4) - (k-4)$$

$$X(G, k) = k^4 - 16k^3 + 50k^2 - 80k + 270$$

In order for any coloring to be consistent with the puzzle, k must be at least as large as the number of distinct colors already used. [3] Therefore, in our example it would be $k \leq 4$. We see that $4 = 1$, which is true because this puzzle has only

one possible completion based on the rules of Sudoku as seen below:

3	1	2	4
4	2	1	3
1	3	4	2
2	4	3	1

Figure 15:

7 Conclusion

Although we were not able to accomplish the greatest goal of proving that their is not a valid initial board up to 16 clues, we were able to get to seven by hand. we also were able to find some patterns within 4x4 boards and explore graph theory's role in Sudoku a little bit. My brain still wonders if there is a way in which we can ultimately prove the rest without a computer. With more time, I would have liked to learn more about the chromatic polynomial and its relationship to Sudoku.

References

- [1] Gary McGuire, Bastian Tugemann, Gilles Civario, 2013: There is no 16-Clue Sudoku: Solving the Sudoku Minimum Number of Clues Problem. [<https://arxiv.org/abs/1201.0749>]
- [2] Reich, Eugenie Samuel, 2012: Mathematician claims breakthrough in Sudoku puzzle. [www.nature.com/news/mathematician-claims-breakthrough-in-sudoku-puzzle-1.9751]
- [3] Oddson, Kyle, 2016: Math and Sudoku. [<http://pdxscholar.library.pdx.edu/studentsymposium/2016/Presentation>]