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Topologically Pure Extensions

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Dedicated to Professor Laszlo Fuchs in honour of his 75th birthday

ABSTRACT. A proper short exact sequence

$$0 \to H \to G \to K \to 0 \quad (*)$$

in the category of locally compact abelian groups is said to be *topologically pure* if the induced sequence

$$0 \to \overline{nH} \to \overline{nG} \to \overline{nK} \to 0$$

is proper short exact for all positive integers n. Some characterizations of topologically pure sequences in terms of direct decompositions, pure extensions and tensor products are established. A simple proof is given for a theorem on pure subgroups by Hartman and Hulanicki. Using topologically pure extensions, we characterize those splitting locally compact abelian groups whose torsion part is a direct sum of a compact group and a discrete group. We determine the compact and discrete groups H with the property that every topologically pure sequence $(*)$ splits. Some structural information on topologically pure injectives and projectives is obtained.

1. Introduction

All groups in this paper are assumed to be Hausdorff abelian topological groups. Let $\mathfrak{L}$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called *proper* if it is open onto its image. An exact sequence

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} G_n$$

in $\mathfrak{L}$ is called *proper exact* if each morphism α_i is proper. A proper short exact sequence $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 0$ in $\mathfrak{L}$ is called an extension of H by K (in $\mathfrak{L}$), and H may be identified with $\alpha(H)$ and K with $G/\alpha(H)$. The group of extensions of H by K is denoted by $\text{Ext}(K, H)$, and $\text{Pext}(K, H)$ is the subgroup of pure extensions of H by K.

Recall that a proper exact sequence $0 \to H \to G \to K \to 0$ splits exactly if H is a (topological) direct summand of G. If G is a direct sum of closed subgroups A and B, then we write $G = A \oplus B$. We say that G *splits* if the torsion part tG of G

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is a direct summand of G. For example, G splits if tG is locally compact and G/tG is compact, which can be derived from [**1**] (6.28). Note that if tG is finite, then G need not split (cf. [**8**] or Example 2.4).

In [**4**], Fulp studied pure exact sequences in the category $\mathfrak{L}$. As it was pointed out by Armacost [**1**], much of the paper is based on [**4**] Proposition 2 (stating that the dual sequence of a proper pure exact sequence is pure) which is unfortunately not true for all extensions in $\mathfrak{L}$. A closed subgroup H of G is said to be *topologically pure* if

$$\overline{nH} = H \cap \overline{nG}$$

for every positive integer n (cf. [**12**]). A pure subgroup need not be topologically pure, and a topologically pure subgroup need not be pure. The identity component G_0 and the subgroup bG of all compact elements of an LCA group G are both pure and topologically pure. It turns out that a closed subgroup H of an LCA group G is pure if and only if its annihilator is topologically pure in the dual group and each subgroup $G[n]+H$ is closed in G (cf. Proposition 2.1). Using this result, we give a short proof of the following theorem which is due to Hartman and Hulanicki [**6**]: If G is an LCA group which is either compactly generated or has no small subgroups, then a closed subgroup of G is pure if and only if its annihilator is pure in the dual group (Theorem 2.3).

Some characterizations of topologically pure sequences in terms of direct decompositions, pure exact sequences and tensor products are contained in Theorem 2.5 and Proposition 2.7. We prove that a topologically pure exact sequence $0 \to B \to G \to K \to 0$ splits if $B = C \oplus D$ is a bounded group where C is compact and D is discrete (Theorem 3.3). Notice that this result generalizes [**9**, Theorem 1.3] and [**10**, Theorem 2.4].

In Example 3.6, a non-splitting topologically pure exact sequence $0 \to B \to G \to K \to 0$ with bounded group B is constructed. It is shown that an LCA group G whose torsion part is a direct sum of a compact group and a discrete group, splits if and only if it possesses an ascending sequence $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ $(n < \omega)$ of open subgroups such that (i) $\bigcup_{n<\omega} A_n = G$, (ii) the torsion part of A_n is bounded for each $n < \omega$, (iii) $t(G/A_n) = (tG + A_n)/A_n$ for each $n < \omega$ and (iv) $0 \to tA_n \to A_n \to A_n/tA_n \to 0$ is a topologically pure exact sequence for each $n \leq \omega$ where $A_\omega = G$ (Theorem 3.7). Dually, we obtain a characterization of a certain class of LCA groups G whose subgroup $\text{div}G = \bigcap_{n<\omega} \overline{nG}$ is a direct summand (Theorem 3.8).

If an LCA group has the injective property relative to the class of topologically pure sequences, then its identity component is a direct sum of a vector group and a toral group. It is shown that the groups of the form $R \oplus T \oplus A \oplus B$ (where R is a vector group, T is a toral group, A is a topological direct product of finite cyclic groups and B is a discrete bounded group) are exactly those LCA groups G for which G/G_0 has a pure compact open subgroup and every topologically pure sequence $0 \to G \to X \to Y \to 0$ splits (cf. Theorem 4.3). In particular: A discrete group B has the property that that every topologically pure sequence starting with B splits if and only if B is bounded, and the compact groups C with the property that every topologically pure sequence starting with C splits, are exactly the groups of the form $T \oplus A$ where T is a toral group and A is a topological direct product of finite cyclic groups (see Theorem 4.1). If an LCA group has the projective property relative to the class of topologically pure sequences, then it has the form

$R \oplus F \oplus bG$ where R is a vector group and F is a free group. The projective groups in $\mathfrak{L}$ are precisely those LCA groups G with the property that every topologically pure sequence $0 \to X \to Y \to G \to 0$ splits and bG possesses a topologically pure compact open subgroup (see Theorem 4.4).

The additive topological group of real numbers is denoted by $\mathbf{R}$, $\mathbf{Q}$ is the group of rationals and $\mathbf{Z}$ is the group of integers. By $\mathbf{T}$ we mean the quotient $\mathbf{R}/\mathbf{Z}$, $\mathbf{Z}(n)$ is the cyclic group of order n and $\mathbf{Z}(p^\infty)$ denotes the quasicyclic group. For any groups G and H, $\mathrm{Hom}(G, H)$ is the group of all continuous homomorphisms from G to H, endowed with the compact-open topology. The dual group of G is

$$\hat{G} = \mathrm{Hom}(G, \mathbf{T})$$

and $(\hat{G}, S)$ denotes the annihilator of $S \subset G$ in $\hat{G}$. Throughout this paper, we use the term "isomorphic" for "topologically isomorphic", and "direct summand" for "topological direct summand". For details and fundamental results on locally compact abelian groups and Pontrjagin duality, we may refer to the book [7] by Hewitt and Ross.

2. Pure and topologically pure extensions

The annihilator of a closed pure subgroup of an LCA group G is topologically pure in the dual group if G is a topological p-group (cf. [12] Theorem 22). This restriction of G is redundant:

PROPOSITION 2.1. *Let H be a closed subgroup of $G \in \mathfrak{L}$. Then we have:*

1. *If H is pure in G, then $(\hat{G}, H)$ is topologically pure in $\hat{G}$.*
2. *If H is topologically pure in G such that $\hat{G}[n] + (\hat{G}, H)$ is a closed subgroup of $\hat{G}$ for all positive integers n, then $(\hat{G}, H)$ is pure in $\hat{G}$.*

PROOF. If H is pure in G, then $(G/H)[n] = (G[n] + H)/H$ for all positive integers n by [3] Theorem 28.1. Let $\varphi : G \to G/H$ be the natural map and let $\rho : (G/H)\hat{} \to (\hat{G}, H)$ be the induced topological isomorphism. Then ρ maps each group $((G/H)\hat{}, (G/H)[n])$ onto $\overline{n(\hat{G}, H)}$. On the other hand, ρ maps $((G/H)\hat{}, (G[n] + H)/H)$ onto $(\hat{G}, G[n] + H)$ which is the same as $\overline{n\hat{G}} \cap (\hat{G}, H)$. Therefore $(\hat{G}, H)$ is topologically pure in $\hat{G}$. The proof of the second assertion is similar. □

COROLLARY 2.2. *Suppose H is a compact or an open subgroup of $G \in \mathfrak{L}$. Then H is pure in G if and only if $(\hat{G}, H)$ is topologically pure in $\hat{G}$.*

A topological group is said to have *no small subgroups* if there exists a neighborhood of 0 which does not contain any nontrivial subgroups. Moskowitz [11] proved that LCA groups without small subgroups have the form $\mathbf{R}^n \oplus \mathbf{T}^m \oplus D$ where m and n are nonnegative integers and D is discrete, and that their Pontrjagin duals are exactly the compactly generated LCA groups. Let $\mathfrak{K}$ denote the class consisting of all LCA groups which are either compactly generated or have no small subgroups. Let G be in $\mathfrak{K}$ and suppose H is a closed subgroup of G. Then H is in $\mathfrak{K}$ as well (see [11] Theorem 2.6). But then, H is pure if and only if H is topologically pure. Therefore, the following result of Hartman and Hulanicki [6] is an immediate consequence of Proposition 2.1:

THEOREM 2.3. [6, Hartman and Hulanicki] *Suppose G is an LCA group which is either compactly generated or has no small subgroups. If H is a closed subgroup of G, then H is pure in G if and only if $(\hat{G}, H)$ is pure in $\hat{G}$.*

Recall that a proper exact sequence $E: 0 \to H \to G \to K \to 0$ in $\mathfrak{L}$ is said to be *pure* if the sequence

$$0 \to nH \to nG \to nK \to 0$$

is exact for all positive integers n. We call the sequence E *topologically pure* if

$$0 \to \overline{nH} \to \overline{nG} \to \overline{nK} \to 0$$

is a proper exact sequence for all positive integers n.

EXAMPLE 2.4. Let p be a prime and n a positive integer, and let H be any densely divisible LCA group such that $H/p^nH \neq 0$. [For instance, topologize the group $A = \prod_{\aleph_0} \mathbf{Z}(p^\infty)$ so that it is locally compact and contains the topological direct product $A[p] = \prod_{\aleph_0} \mathbf{Z}(p)$ as an open subgroup, and set $H = A[p] + \bigoplus_{\aleph_0} \mathbf{Z}(p^\infty) \subset A$ (cf. [8]).] Then $\mathrm{Ext}(\mathbf{Z}(p^n), H) \neq 0$, so there is a non-splitting extension

$$E_1 : 0 \to H \to G \to \mathbf{Z}(p^n) \to 0$$

in $\mathfrak{L}$. Notice that E_1 is topologically pure but not pure, and that the dual sequence

$$E_2 : 0 \to \mathbf{Z}(p^n) \to \hat{G} \to \hat{H} \to 0$$

is pure because $\hat{H}$ is torsion-free, but not topologically pure.

Some characterizations of topologically pure sequences are contained in the next theorem.

THEOREM 2.5. *Consider the following conditions for a proper exact sequence* $E: 0 \to H \to G \to K \to 0$ *in* $\mathfrak{L}$*:*

1. $0 \to \overline{nH} \to \overline{nG} \to \overline{nK} \to 0$ *is proper exact for all* n*;*
2. $0 \to H/\overline{nH} \to G/\overline{nG} \to K/\overline{nK} \to 0$ *is proper exact for all* n*;*
3. $\overline{nG + H}/\overline{nH} = \overline{nG}/\overline{nH} \oplus H/\overline{nH}$ *for all* n*;*
4. $0 \to H[n] \to G[n] \to K[n] \to 0$ *is proper exact for all* n*;*
5. $0 \to H/H[n] \to G/G[n] \to K/K[n] \to 0$ *is proper exact for all* n*;*
6. $(n^{-1}H)/H[n] = G[n]/H[n] \oplus H/H[n]$ *for all* n*.*

Then we have: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) *and* (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). *Further,* (1) $\not\Rightarrow$ (4) $\not\Rightarrow$ (1) *generally.*

PROOF. Suppose the sequence E is topologically pure. Then $\overline{nG} + H$ is a closed subgroup of G, and the topological isomorphism $(\overline{nG} + H)/H \to \overline{nG}/\overline{nH}$ induces a continuous homomorphism

$$\phi : (\overline{nG} + H)/\overline{nH} \to \overline{nG}/\overline{nH}$$

which is the identity on $\overline{nG}/\overline{nH}$ so that the kernel of ϕ is equal to $H/\overline{nH}$. By [7] (6.22), $(\overline{nG} + H)/\overline{nH}$ is a direct sum of $\overline{nG}/\overline{nH}$ and $H/\overline{nH}$, and the equivalence of (1) and (3) follows. Next we assume that $0 \to H[n] \to G[n] \to K[n] \to 0$ is proper exact. Then there is a continuous homomorphism

$$\psi : (n^{-1}H)/H[n] \to G[n]/H[n]$$

which is the identity on $G[n]/H[n]$ so that $\ker \psi = H/H[n]$, hence $(n^{-1}H)/H[n]$ is a direct sum of $G[n]/H[n]$ and $H/H[n]$. Consequently, (4) and (6) are equivalent. To prove (2) $\Leftrightarrow$ (3) and (5) $\Leftrightarrow$ (6), similar arguments can be used. Finally, Example 2.4 shows that (1) $\not\Rightarrow$ (4) $\not\Rightarrow$ (1). □

Let us call a proper exact sequence $0 \to H \to G \to K \to 0$ in $\mathfrak{L}$ $*$*-pure* if it satisfies the (equivalent) conditions (4), (5) and (6) in Theorem 2.5. Then we have:

COROLLARY 2.6. *A proper exact sequence in $\mathfrak{L}$ is $*$-pure if and only if its dual sequence is topologically pure.*

PROOF. The sequence $0 \to H[n] \to G[n] \to K[n] \to 0$ is proper exact if and only if the dual sequence

$$0 \to \hat{K}/\overline{n\hat{K}} \to \hat{G}/\overline{n\hat{G}} \to \hat{H}/\overline{n\hat{H}} \to 0$$

is proper exact. □

Following Fulp [4], we define the tensor product of LCA groups G and H to be the topological group

$$G \otimes H = \operatorname{Hom}(G, \hat{H})\hat{}.$$

Note that $G \otimes H$ is locally compact if G is finitely generated, and in this case, our definition coincides with the definition of Moskowitz in [11]. If G and H are discrete, then $G \otimes H$ is the usual tensor product of discrete abelian groups.

PROPOSITION 2.7. *Suppose $E : 0 \to H \to G \to K \to 0$ is a proper exact sequence in $\mathfrak{L}$ such that $\hat{G}$ is σ-compact. Then the following conditions are equivalent:*

1. *E is a topologically pure sequence;*
2. *the sequence $0 \to F \otimes H \to F \otimes G$ is proper exact for every finitely generated discrete group F;*
3. *the sequence $0 \to \mathbf{Z}(n) \otimes H \to \mathbf{Z}(n) \otimes G$ is proper exact for every positive integer n.*

PROOF. (1) implies (2). For any LCA group G, we write G_d for the group G with the discrete topology. Suppose $0 \to H \to G \to K \to 0$ is topologically pure and let F be a finitely generated discrete group. Then the sequence $0 \to \hat{K}_d \to \hat{G}_d \to \hat{H}_d \to 0$ is pure by Corollary 2.6, hence

$$0 \to \operatorname{Hom}(F, \hat{K}_d) \to \operatorname{Hom}(F, \hat{G}_d) \to \operatorname{Hom}(F, \hat{H}_d) \to \operatorname{Pext}(F, \hat{K}_d) = 0$$

is an exact sequence. Notice that $\operatorname{Hom}(F, \hat{G})$ is σ-compact, hence the sequence

$$\operatorname{Hom}(F, \hat{G}) \to \operatorname{Hom}(F, \hat{H}) \to 0$$

is proper exact by the Open Mapping Theorem. Consequently, the sequence in (2) is proper exact.

It is clear that (2) implies (3). Now suppose $0 \to \mathbf{Z}(n) \otimes H \to \mathbf{Z}(n) \otimes G$ is proper exact for all n. Then each induced homomorphism $\operatorname{Hom}(\mathbf{Z}(n), \hat{G}) \to \operatorname{Hom}(\mathbf{Z}(n), \hat{H})$ is surjective. Since $\hat{G}$ is σ-compact, it follows that $0 \to \hat{K} \to \hat{G} \to \hat{H} \to 0$ is $*$-pure. By Corollary 2.6, the sequence E is topologically pure. Therefore, (3) implies (1). □

Notice that Proposition 2.7 fails if "topologically pure" is replaced by "pure" (cf. [4] Proposition 3): For instance, the group G in Example 2.4 possesses a sequence of compact open subgroups H_i whose intersection is trivial, hence

$$\hat{G} = \sum(\hat{G}, H_i)$$

is σ-compact. The tensor map $F \otimes H \to F \otimes G$ is proper and injective for every finitely generated group F, but H is not pure in G.

3. Extensions of torsion groups

A pure subgroup H of a discrete group G is a direct summand if the quotient G/H is a direct sum of cyclic groups (cf. [3] Theorem 28.2). By Corollary 2.2 and duality, we have:

THEOREM 3.1. *Let H be a closed topologically pure subgroup of an LCA group G. Then H is a direct summand if it has the form $\mathbf{T}^{\mathfrak{m}} \oplus A$, where $\mathfrak{m}$ is a cardinal and A is a topological direct product of finite cyclic groups. In particular, a compact torsion subgroup of an LCA group is a direct summand if it is topologically pure.*

PROPOSITION 3.2. *Let $0 \to H \to G \to K \to 0$ be a proper exact sequence in $\mathfrak{L}$ and suppose H is a discrete group which is a direct sum of cyclic groups of the same order p^k where p is a prime and k is a positive integer. Then the following conditions are equivalent:*

1. *$0 \to H \to G \to K \to 0$ is topologically pure;*
2. *$\overline{p^kG + H} = \overline{p^kG} \oplus H$;*
3. *$0 \to H \to G \to K \to 0$ splits.*

PROOF. (1) implies (2) because of Theorem 2.4.

(2) implies (3). Suppose $\overline{p^kG + H} = \overline{p^kG} \oplus H$. Since $G/\overline{p^kG}$ is totally disconnected, it contains a compact open subgroup $C/\overline{p^kG}$ such that $C \cap H = 0$. The set of all open subgroups A of G such that $A \cap H = 0$ and $p^kG \subset A$ is partially ordered by inclusion and contains a maximal element K by Zorn's lemma. Now it can be shown that the quotient $G/(H \oplus K)$ is both torsion and torsion-free (see the proof of [3] Proposition 27.1), thus $0 \to H \to G \to K \to 0$ splits.

It is clear that (3) implies (1). □

Recall that a bounded pure subgroup of a discrete group is a direct summand (see [3] Theorem 27.5). This result can be extended as follows:

THEOREM 3.3. *Suppose B is a bounded group which is a direct sum of a compact group and a discrete group. Then:*

1. *A proper exact sequence $0 \to B \to G \to K \to 0$ in $\mathfrak{L}$ splits if and only if it is topologically pure.*
2. *Dually, a proper exact sequence $0 \to H \to G \to B \to 0$ in $\mathfrak{L}$ splits if and only if it is $*$-pure.*

PROOF. By Corollary 2.6 and duality, it suffices to prove the first assertion. Suppose $0 \to B \to G \to K \to 0$ is a topologically pure exact sequence where B is a bounded group.

First, we assume that B is discrete and write $B = H \oplus H'$ where H is a direct sum of cyclic groups of the same prime power p^k and the maximum of orders of the elements of H' is less than the maximum of orders of the elements of B. By Theorem 2.5, we have

$$\overline{p^kG + B}/\overline{p^kB} = \overline{p^kG}/\overline{p^kB} \oplus B/\overline{p^kB},$$

therefore $\overline{p^kG}$ is an open subgroup of $\overline{p^kG + H}$. Now Proposition 3.2 shows that there is a closed subgroup G' of G such that $G = H \oplus G'$. Clearly, we have $B = H \oplus (G' \cap B)$. The sequence

$$0 \to G' \cap B \to G' \to G'/G' \cap B \to 0$$

is topologically pure and splits by induction. Therefore, $0 \to B \to G \to K \to 0$ splits.

Now suppose that B is a direct sum of a compact group C and a discrete group D. By Theorem 3.1, we can write $G = C \oplus X$ and by the first part of this proof, $X \cap B$ is a direct summand of X. Again, the sequence $0 \to B \to G \to K \to 0$ splits. The converse is obvious. □

COROLLARY 3.4. *Let B be a bounded LCA group consisting of elements of square-free order. Then a proper exact sequence $0 \to B \to G \to K \to 0$ in $\mathfrak{L}$ splits if it is topologically pure.*

A discrete bounded topologically pure subgroup of an LCA group need not be a direct summand:

EXAMPLE 3.5. Let p be a prime and let A be any proper dense subgroup (taken discrete) of the topological direct product $H = \prod_{\aleph_0} \mathbf{Z}(p^2)$. Then $B = \{(px, px) : x \in A\}$ is a discrete subgroup of $G = A \times pH$. We have $\overline{pB} = B \cap \overline{pG}$, therefore B is topologically pure in G. Since $\overline{pG} + B$ is not a closed subgroup of G, B is not a direct summand of G.

A topologically pure exact sequence $0 \to B \to G \to K \to 0$ with bounded group B need not split:

EXAMPLE 3.6. Take the locally compact group $A = \prod_{\aleph_0} \mathbf{Z}(p^\infty)$ of Example 2.4, let G be the subgroup $A[p^2]$ and consider the subgroup B consisting of all elements $(x_i) \in G = \prod_{\aleph_0} \mathbf{Z}(p^2)$ such that $px_i = 0$ for almost all i. Then $\overline{p(G/B)} = 0$ and therefore the sequence $0 \to B \to G \to G/B \to 0$ is topologically pure. Since the sequence is not pure, it does not split.

Splitting LCA groups whose torsion part is a direct sum of a compact group and a discrete group, can be characterized using topologically pure extensions:

THEOREM 3.7. *Let G be an LCA group so that tG is a direct sum of a compact group and a discrete group. Then G splits if and only if it possesses an ascending sequence $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ $(n < \omega)$ of open subgroups such that*

1. *$\bigcup_{n<\omega} A_n = G$;*
2. *tA_n is bounded for all $n < \omega$;*
3. *$t(G/A_n) = (tG + A_n)/A_n$ for all $n < \omega$;*
4. *$0 \to tA_n \to A_n \to A_n/tA_n \to 0$ is a topologically pure sequence for all $n \le \omega$ (we set $A_\omega = G$).*

PROOF. Suppose $tG = C \oplus D$ where C is compact and D is discrete. If $G = tG \oplus F$, then the open subgroups $A_n = C \oplus D[n!] \oplus F$ satisfy conditions (1)-(4).

Conversely, assume that G has open subgroups A_n satisfying (1) - (4). By Theorem 3.1, there is a closed subgroup G' of G such that $G = C \oplus G'$. Since C is compact, we may assume that $C \subset A_n$ for each n. Letting $A'_n = A_n \cap G'$ we have $A_n = C \oplus A'_n$. By Theorem 3.3, we can write $A_n = tA_n \oplus H_n$, hence

$$A'_n = (tA_n \oplus H_n) \cap G' = (tA'_n \oplus C \oplus H_n) \cap G' = tA'_n \oplus [(C \oplus H_n) \cap G'].$$

Since $t(G'/A'_n) = (tG' + A'_n)/A'_n$ for all n, the groups $G_n = (C \oplus H_n) \cap G'$ can be chosen so that $G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots$ (see the proof of [3] Proposition 100.4). Consequently, we have $G = C \oplus (D \oplus \bigcup_{n<\omega} G_n) = tG \oplus \bigcup_{n<\omega} G_n$, as desired. □

For an LCA group G, a closed subgroup div$G$ is defined by

$$\mathrm{div}G = \bigcap_{n<\omega} \overline{nG}.$$

Then dualization of Theorem 3.7 yields a characterization of a certain class of LCA groups G whose subgroup div$G$ is a direct summand:

THEOREM 3.8. *Let G be an LCA group such that $G/\mathrm{div}G$ is a direct sum of a compact totally disconnected group and a discrete bounded group. Then* div$G$ *is a direct summand of G if and only if G has a descending sequence $B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots$ $(n < \omega)$ of compact subgroups such that*

1. $\bigcap_{n<\omega} B_n = 0$;
2. $G/(\mathrm{div}G + B_n)$ *is bounded for all* $n < \omega$;
3. $\mathrm{div}B_n = \mathrm{div}G \cap B_n$ *for all* $n < \omega$;
4. $0 \to (\mathrm{div}G + B_n)/B_n \to G/B_n \to G/(\mathrm{div}G + B_n) \to 0$ *is a $*$-pure sequence for all* $n \leq \omega$ *(we set* $B_\omega = 0$*).*

PROOF. The subgroup div$G$ is a direct summand of G exactly if $t\hat{G} = (\hat{G}, \mathrm{div}G)$ is a direct summand of $\hat{G}$. The sequence

$$B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots$$

satisfies the conditions (1) - (4) in Theorem 3.8 if and only if the sequence

$$(\hat{G}, B_1) \subset (\hat{G}, B_2) \subset \ldots \subset (\hat{G}, B_n) \subset \ldots$$

satisfies the conditions (1) - (4) in Theorem 3.7. □

4. Splitting problems

Let G and X be groups in $\mathfrak{L}$. Recall that $\mathrm{Pext}(X, G) = 0$ if and only if every pure exact sequence $0 \to G \to H \to X \to 0$ splits. We say that

$${}^*\mathrm{Pext}(X, G) = 0$$

if every $*$-pure sequence $0 \to G \to H \to X \to 0$ splits. Similarly, we say that

$${}^t\mathrm{Pext}(X, G) = 0$$

if every topologically pure sequence $0 \to G \to H \to X \to 0$ splits.

THEOREM 4.1. *Let G be an LCA group.*

1. *If G is discrete, then* ${}^*\mathrm{Pext}(X, G) = 0$ *for each X in $\mathfrak{L}$ if and only if $G = 0$.*
2. *If G is discrete, then* ${}^t\mathrm{Pext}(X, G) = 0$ *for each X in $\mathfrak{L}$ if and only if G is bounded.*
3. *If G is compact, then* ${}^*\mathrm{Pext}(X, G) = 0$ *for each X in $\mathfrak{L}$ if and only if* $G \cong \mathbf{T}^{\mathfrak{m}}$ *where $\mathfrak{m}$ is a cardinal.*
4. *If G is compact, then* ${}^t\mathrm{Pext}(X, G) = 0$ *for each X in $\mathfrak{L}$ if and only if* $G \cong \mathbf{T}^{\mathfrak{m}} \oplus A$ *where $\mathfrak{m}$ is a cardinal and A is a topological direct product of finite cyclic groups.*

PROOF. (1) Suppose G is discrete such that ${}^*\mathrm{Pext}(X, G) = 0$ for each X in $\mathfrak{L}$. Since any extension of G by $\hat{\mathbf{Q}}$ is $*$-pure, we have $\mathrm{Ext}(\hat{\mathbf{Q}}, G) = 0$. Hence the exactness of

$$\mathrm{Ext}(\hat{\mathbf{Q}}, tG) \to \mathrm{Ext}(\hat{\mathbf{Q}}, G) \to \mathrm{Ext}(\hat{\mathbf{Q}}, G/tG) \to 0$$

implies that $\operatorname{Ext}(\hat{\mathbf{Q}}, G/tG) = 0$. The sequence

$$\operatorname{Hom}((\mathbf{Q}/\mathbf{Z})\hat{}, G/tG) \to \operatorname{Ext}(\hat{\mathbf{Z}}, G/tG) \to \operatorname{Ext}(\hat{\mathbf{Q}}, G/tG)$$

is exact and $\operatorname{Hom}((\mathbf{Q}/\mathbf{Z})\hat{}, G/tG) = 0$. Therefore we have

$$G/tG \cong \operatorname{Ext}(\mathbf{T}, G/tG) = 0,$$

hence G is a torsion group. Since G is also a cotorsion group, it is a direct sum of a bounded group B and a divisible group D (cf. [**3**] Corollary 54.4). Consequently, ${}^*\operatorname{Pext}(X, B) = 0$ and ${}^*\operatorname{Pext}(X, D) = 0$ for each X in $\mathfrak{L}$. Since every nontrivial bounded discrete group can be identified with the torsion part of some non-splitting LCA group (see Example 2.4), $B = 0$ follows. Further, ${}^*\operatorname{Pext}(X, Q) = \operatorname{Ext}(X, Q)$ if Q is quasicyclic, therefore $D = 0$. This proves the first statement.

(2) Now suppose G is discrete such that ${}^t\operatorname{Pext}(X, G) = 0$ for each X in $\mathfrak{L}$. Then $\operatorname{Ext}(\hat{\mathbf{Q}}, G) = 0$ and again, we conclude that G is a reduced torsion group. Since G is also algebraically compact, G is bounded (cf. [**3**] Corollary 40.3). The converse is true because of Theorem 3.3.

(3) If G is compact where ${}^*\operatorname{Pext}(X, G) = 0$ for each X in $\mathfrak{L}$, then $\operatorname{Pext}(\hat{G}, C) = 0$ for all discrete groups C, hence $\hat{G}$ is a direct sum of cyclic groups. Consequently, G is of the form $\mathbf{T}^{\mathfrak{m}} \oplus A$ where A is a topological direct product of finite cyclic groups. By (1), A is trivial. Since $\operatorname{Ext}(X, \mathbf{T}^{\mathfrak{m}}) = 0$ for all X in $\mathfrak{L}$, the third assertion follows.

(4) The last statement follows from the proof of (3) and Theorem 3.1. □

Dually, we have:

THEOREM 4.2. *Let G be an LCA group.*

1. *If G is discrete, then ${}^*\operatorname{Pext}(G, X) = 0$ for each X in $\mathfrak{L}$ if and only if G is a direct sum of cyclic groups.*
2. *If G is discrete, then ${}^t\operatorname{Pext}(G, X) = 0$ for each X in $\mathfrak{L}$ if and only if G is free.*
3. *If G is compact, then ${}^*\operatorname{Pext}(G, X) = 0$ for each X in $\mathfrak{L}$ if and only if G is torsion.*
4. *If G is compact, then ${}^t\operatorname{Pext}(G, X) = 0$ for each X in $\mathfrak{L}$ if and only if $G = 0$.*

The next theorem contains some structural information on those LCA groups G with the property that every $*$-pure (resp. topologically pure) exact sequence $0 \to G \to H \to X \to 0$ splits.

THEOREM 4.3. *Let G be an LCA group.*

1. *If for each group X in $\mathfrak{L}$, ${}^*\operatorname{Pext}(X, G) = 0$ or ${}^t\operatorname{Pext}(X, G) = 0$, then $G \cong \mathbf{R}^n \oplus \mathbf{T}^{\mathfrak{m}} \oplus G'$ where G' is totally disconnected.*
2. *${}^*\operatorname{Pext}(X, G) = 0$ for each X in $\mathfrak{L}$ and G/G_0 possesses a pure compact open subgroup if and only if $G \cong \mathbf{R}^n \oplus \mathbf{T}^{\mathfrak{m}}$.*
3. *${}^t\operatorname{Pext}(X, G) = 0$ for each X in $\mathfrak{L}$ and G/G_0 possesses a pure compact open subgroup if and only if $G \cong \mathbf{R}^n \oplus \mathbf{T}^{\mathfrak{m}} \oplus A \oplus B$, where A is a topological direct product of finite cyclic groups and B is a discrete bounded group.*

PROOF. (1) Let C be a connected LCA group. Then the proper exact sequence $0 \to G_0 \xrightarrow{\alpha} G \to G/G_0 \to 0$ induces the exact sequence

$$0 = \operatorname{Hom}(C, G/G_0) \to \operatorname{Ext}(C, G_0) \xrightarrow{\alpha_*} \operatorname{Ext}(C, G).$$

If $E: 0 \to G_0 \xrightarrow{\phi_1} X \to C \to 0 \in \mathrm{Ext}(C, G_0)$, then we have

$$\alpha_*(E) = \alpha E : 0 \to G \xrightarrow{\phi_2} X' \to C \to 0$$

where $X' = (G \oplus X)/N$ and $N = \{(-\alpha(g), \phi_1(g)) : g \in G_0\}$. Since G_0 and C are connected, X is connected (cf. [**7**] (7.14)), hence $X[m]$ is a compact group for each positive integer m. But then $X'[m]/\phi_2(G[m])$ is compact as well which implies that the sequence

$$0 \to G[m] \to X'[m] \to C[m] \to 0$$

is proper exact. Therefore, αE is $*$-pure. Since we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & G_0 & \to & X & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \overline{mG} & \to & \overline{mX'} & \to & \overline{mC} & \to & 0 \end{array}$$

with proper exact bottom row, αE is also topologically pure. It follows that $\mathrm{Ext}(C, G_0) = 0$ if ${}^*\mathrm{Pext}(C, G) = 0$ or ${}^t\mathrm{Pext}(C, G) = 0$. By [**5**] Theorem 3.3, the identity component of G is isomorphic to $\mathbf{R}^n \oplus \mathbf{T}^m$ and is therefore a direct summand of G (cf. [**11**] Theorem 3.2).

(2) Suppose ${}^*\mathrm{Pext}(X, G) = 0$ for each X in $\mathfrak{L}$ such that $D = G/G_0$ has a pure compact open subgroup A. By (1), G_0 is a direct summand of G, therefore A is an algebraically compact open subgroup of D. Then there is a discrete subgroup B of G so that $D = A \oplus B$. It follows that ${}^*\mathrm{Pext}(X, A) = 0$ and ${}^*\mathrm{Pext}(X, B) = 0$ for each X in $\mathfrak{L}$ and Theorem 4.1 yields $A = B = 0$.

(3) Finally, assume that ${}^t\mathrm{Pext}(X, G) = 0$ for all X in $\mathfrak{L}$ such that $D = G/G_0$ has a pure compact open subgroup A. Again, we can write $D = A \oplus B$ and by Theorem 4.1, A is a topological direct product of finite cyclic groups and B is a discrete bounded group. Conversely, every group G of the form $\mathbf{R}^n \oplus \mathbf{T}^m \oplus A \oplus B$ as in (3) satisfies ${}^t\mathrm{Pext}(X, G) = 0$ for all X in $\mathfrak{L}$. This completes the proof of the theorem. □

Dualization of Theorem 4.3 yields the following result involving the subgroup bG of all compact elements of G:

THEOREM 4.4. *Let G be an LCA group.*

1. *If for each group X in $\mathfrak{L}$, ${}^*\mathrm{Pext}(G, X) = 0$ or ${}^t\mathrm{Pext}(G, X) = 0$, then $G \cong \mathbf{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbf{Z} \oplus bG$.*
2. *${}^*\mathrm{Pext}(G, X) = 0$ for each X in $\mathfrak{L}$ and bG possesses a topologically pure compact open subgroup if and only if $G \cong \mathbf{R}^n \oplus C \oplus D$ where C is a compact torsion group and D is a discrete group which is a direct sum of cyclic groups.*
3. *${}^t\mathrm{Pext}(G, X) = 0$ for each X in $\mathfrak{L}$ and bG possesses a topologically pure compact open subgroup if and only if $G \cong \mathbf{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbf{Z}$.*

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