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Abstract. Prime ideals of strong semigroup graded rings have been characterized by Bell, Stalder and Teply for some important classes of semigroups. The prime ideals correspond to certain families of ideals of the component rings called prime families. In this paper, we define the notion of a primitive family and show that primitive ideals of such rings correspond to primitive families of ideals of the component rings.

0. Introduction

A semigroup is a set with an associative binary operation. Let Ω be a semigroup. Let $\prec$ be a reflexive and transitive relation on Ω (i.e., a preorder). Assume that (Ω, $\prec$) is directed, that is, for any $\alpha, \beta \in \Omega$, there exists $\gamma \in \Omega$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$. A system of rings $\mathcal{R}$ over $(\Omega, \prec)$ is a collection $(R_\alpha)_{\alpha\in\Omega}$ of rings, together with ring homomorphisms $\phi_{\alpha,\beta} : R_\alpha \to R_\beta$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$ such that $\phi_{\alpha,\alpha} = id_{R_\alpha}$ for all α and $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ whenever $\alpha \prec \beta \prec \gamma$. Throughout this paper, the preorder $\prec$ on Ω is defined by $\alpha \prec \beta$ if β is in the ideal generated by α, that is, β can be written as a product of one or more factors involving α as a factor. This preorder is defined in [1] and is called the ideal preorder. With a system $\mathcal{R}$ we can associate a ring $R = \oplus\mathcal{R}$, a direct sum of R_α's as additive abelian groups, with multiplication in R defined via $r_\alpha \cdot r_\beta = \phi_{\alpha,\alpha\beta}\ (r_\alpha)\phi_{\beta,\alpha\beta}(r_\beta)$ for $r_\alpha \in R_\alpha$ and $r_\beta \in R_\beta$. (Note that by the definition of $\prec$, $\alpha \prec \alpha\beta$ and $\beta \prec \alpha\beta$.) We call R a strong Ω-graded ring. Also note that this definition of a strong grading arises from an analogous notion in semigroup theory and is different from the usual definition of a strong grading. If R is a ring, then a semigroup ring $R[\Omega]$ is defined as the set of all finite sums of the form $\sum_{\alpha\in\Omega} r_\alpha\alpha$, where $r_\alpha \in R$. Addition is defined by adding coefficients of like terms and multiplication is defined via $r_\alpha\alpha \cdot r'_\beta\beta = r_\alpha r'_\beta\alpha\beta$. A semigroup ring, $R[\Omega]$ can be viewed as a strong Ω-graded ring by defining $R_\alpha = R\alpha$ for each $\alpha \in \Omega$ and $\phi_{\alpha,\beta}(r\alpha) = r\beta$. Thus the study of strong Ω-graded rings can be applied to the study of semigroup rings.

Strong semigroup graded rings have been studied for various ring theoretic

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properties in [1], [2], [3], [5], [6], and [10]. When Ω is a semilattice, a strong Ω-graded ring is also known as a strong supplementary semilattice sum of rings (see [9] and [12]). Prime ideals of strong Ω-graded rings were characterized in [1] and [2] for some classes of semigroups and have been recently used in the study of π-regularity of such rings in [6]. The classes of semigroups considered include semigroups such as regular bands, power stationary semigroups, commutative periodic semigroups, and Clifford semigroups as special cases. In particular, it was proved that there exists a bijective correspondence between prime ideals of R and certain families of ideals of the component rings called prime families. In this paper we define, analogously, the notion of a primitive family and show that a prime ideal is primitive if and only if its corresponding prime family is also a primitive family.

1. Preliminaries

Let P be an ideal of a ring R. Then P is said to be prime if for ideals A and B in R, $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$. P is completely prime if $ab \in P$ implies that $a \in P$ or $b \in P$ for elements $a, b \in R$. P is left (right) primitive if it is the annihilator of a simple faithful left (right) R-module. A left (right) R-module is said to be simple if it is isomorphic to R/M for some modular maximal left (right) ideal M of R. A maximal left ideal of R is said to be modular if R^2 is not contained in M (see [8]). Note that a primitive ideal of a ring R is also a prime ideal of R. We shall now define some of the semigroup terms used in this paper. A subsemigroup T of Ω is an ideal if for all $t \in T$ and $s \in S$ we have st, $ts \in T$. The definition of a prime (completely prime) ideal in a semigroup is similar to that of a prime (completely prime) ideal in a ring. A band is a semigroup of idempotents. In a band, the ideal preorder is equivalent to $\alpha \prec \beta \Leftrightarrow \beta\alpha\beta = \beta$. A band is said to be regular if it satisfies the identity $xyxzx = xyzx$. A semilattice is a commutative semigroup of idempotents. In a semilattice, the preorder $\prec$ reduces to $\alpha \prec \beta \Leftrightarrow \beta\alpha = \beta$. A semigroup Ω is said to be power stationary if each $x \in \Omega$ satisfies $x^{n+1} = x^n$ for some n depending on x. Ω is periodic if $x^{n+m} = x^n$ for each $x \in \Omega$, where m, n are positive integers depending on x. For more details on semigroups we refer the reader to [4] and [7].

The notion of a compatible family of ideals over $\mathcal{R}$ was first defined in [1]. In this paper we retain the notation and terminology of [1] and [2]. A family of [left, right, two-sided] ideals over $\mathcal{R}$ is a collection $\mathcal{I} = (I_\alpha)_{\alpha\in\Omega}$ such that each I_α is a [left, right, two-sided] ideal of R_α and $\phi_{\alpha,\beta}(I_\alpha) \subseteq I_\beta$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$. We call the family compatible if $\phi^{-1}_{\alpha,\beta}(I_\beta) = I_\alpha$ for all $\alpha, \beta \in \Omega$ with $\alpha \prec \beta$. As in [1], we partially order families over $\mathcal{R}$ by $\mathcal{I} \subseteq I'$ if $I_\alpha \subseteq I'_\alpha$ for all $\alpha \in \Omega$. We call a compatible family maximal if it is maximal among proper compatible families, that is, maximal among compatible families $\mathcal{I}$ with at least one $I_\alpha \neq R_\alpha$. We say that a proper compatible family $\mathcal{P}$ is a prime family if whenever $\mathcal{P} \supseteq \mathcal{I}\mathcal{J}$ for families of ideals, we have $\mathcal{P} \supseteq \mathcal{I}$ or $\mathcal{P} \supseteq J$.

Let Φ be a completely prime ideal of Ω. Define $I(\Phi)$ to be the additive subgroup of $R = \oplus\mathcal{R}$ generated by the elements r_α with $\alpha \in \Phi$ and $r_\alpha - \phi_{\alpha,\beta}(r_\alpha)$, with

$\alpha, \beta \in \Omega \setminus \Phi$ and $\alpha \prec \beta$, where r_α is an arbitrary element of R_α. By [Lemma 2.4, [1]], $I(\Phi)$ is an ideal of $R = \oplus\mathcal{R}$ and $R/I(\Phi) \cong \varinjlim_{\alpha\in\Omega\setminus\Phi} R_\alpha$. Let $\mathcal{I}$ be a compatible family of [left, right, two-sided] ideals over the system $\mathcal{R}|_{\Omega\setminus\Phi}$. Then the pair $(\Phi, \mathcal{I})$ is called a compatible family and if $\mathcal{I}$ is a prime family, we call the pair $(\Phi, \mathcal{I})$ a prime family. The ideal $I(\Phi, \mathcal{I})$ plays an important role in the characterization of prime ideals of $R = \oplus\mathcal{R}$ and is given by $I(\Phi, \mathcal{I}) = I(\Phi) + \bigcup_{\alpha\in\Omega\setminus\Phi} I_\alpha$. The following result proved in [1] gives a complete description of ideals of $R = \oplus\mathcal{R}$ that contain the ideal $I(\Phi)$.

Proposition 1.1 ([1]). *Let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$ and let Φ be a completely prime ideal of Ω. Then there exists an order preserving bijection between the set of [left, right, two-sided] ideals of $R = \oplus\mathcal{R}$ containing $I(\Phi)$ and the set of compatible families of [left, right, two-sided] ideals given by $I \longmapsto (\Phi, (I \cap R_\alpha)_{\alpha\in\Omega\setminus\Phi})$, with inverse given by $(\Phi, \mathcal{I}) \longmapsto I(\Phi, \mathcal{I})$. Moreover, if I is an ideal of R containing $I(\Phi)$, then $R/I \cong \varinjlim_{\alpha\in\Omega\setminus\Phi} R_\alpha/(I \cap R_\alpha)$.*

Definition 1.2 ([1]). We say that a semigroup Ω satisfies condition (†) if for any prime ideal Φ of Ω and for any α, $\beta \in \Omega \setminus \Phi$, there exists a $\gamma \in \Omega \setminus \Phi$ such that $\gamma'\alpha\gamma'' = \gamma'\beta\gamma''$ for all γ', $\gamma'' \in (\gamma) \setminus \Phi$.

Semigroups satisfying condition (†) were first considered in the description of prime ideals of strong Ω-graded rings in [1]. Such semigroups are generalizations of regular or finite bands and power stationary semigroups. Let P be a prime ideal of $R = \oplus\mathcal{R}$. By Lemma 2.6, [1], if Ω satisfies condition (†) and all prime ideals of Ω are completely prime then $P \supseteq I(\Phi)$. In view of Proposition 1.1 the following result on prime ideals was obtained in [1].

Theorem 1.3 ([1]). *Let Ω be a semigroup satisfying condition (†) and Φ be a completely prime ideal of Ω. Let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$. Then there exists an order preserving bijection between the set of prime ideals P in $R = \oplus\mathcal{R}$ and the set of prime families $(\Phi, \mathcal{P})$ with Φ a prime ideal of Ω, given by $P \longmapsto (\Phi = \{\alpha \in \Omega | R_\beta \subseteq P \text{ for all } \beta \in (\alpha)\}, (P \cap R_\alpha)_{\alpha\in\Omega\setminus\Phi})$, with inverse given by $(\Phi, \mathcal{P}) \longmapsto I(\Phi, \mathcal{P})$.*

This result was generalized to some other classes of strong semigroup graded rings in [2].

2. Primitive ideals of strong semigroup graded rings

In this section we begin our study of primitive ideals of those strong Ω-graded rings whose prime ideals were characterized in [1] and [2].

Definition 2.1. Let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$ and Φ be a completely prime ideal of Ω. Then a family of ideals $(\Phi, \mathcal{P})$ is a (left) primitive family if the following conditions are satisfied:

(i) $(\Phi, \mathcal{P})$ is a prime family and

(ii) there exists a maximal compatible family of left ideals $(\Phi, \mathcal{M})$ such that $\mathcal{P} \subseteq \mathcal{M}$ (that is, $P_\alpha \subseteq M_\alpha$ for all $\alpha \in \Omega \setminus \Phi$, $P_\alpha \in \mathcal{P}$ and $M_\alpha \in \mathcal{M}$) and for every $r_\alpha \in R_\alpha \setminus P_\alpha, \alpha \in \Omega \setminus \Phi$ there exists an $s_\gamma \in R_\gamma$ for some $\gamma \in \Omega \setminus \Phi$ such that $r_\alpha \cdot s_\gamma \notin M_{\alpha\gamma}$.

A (right) primitive family is defined similarly. As in the case of prime ideals, if the defining maps $\phi_{\alpha,\beta}$ are all onto then a primitive family consists entirely of primitive ideals. We prove this fact in the next result.

Proposition 2.2. *Let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$. If the defining maps $\phi_{\alpha,\beta}$ for $\alpha \prec \beta$ are all onto, then a primitive family $(\Phi, \mathcal{P})$, where $\mathcal{P} = \{P_\alpha\}_{\alpha\in\Omega\setminus\Phi}$ consists entirely of primitive ideals.*

Proof. Since $(\Phi, \mathcal{P})$ is a primitive family, there exists a maximal compatible family of left ideals $(\Phi, \mathcal{M})$ satisfying condition (ii) in Definition 2.1. Since the maps $\phi_{\alpha,\beta}$ are all onto, $\mathcal{M}$ consists entirely of maximal left ideals and $\mathcal{P}$ consists entirely of prime ideals (see Corollary 1.7 in [1]). Now suppose that $P_\alpha \in \mathcal{P}$ is not primitive for some $\alpha \in \Omega\setminus\Phi$. Then $P_\alpha \neq R_\alpha$ since P_α is prime and $P_\alpha \subset Q_\alpha$, where $Q_\alpha = ann\, R_\alpha/M_\alpha$. Now let $r_\alpha \in Q_\alpha \setminus P_\alpha$. By condition (ii) in the definition of a primitive family, there exists an $s_\gamma \in R_\gamma$ for some $\gamma \in \Omega \setminus \Phi$ such that $r_\alpha \cdot s_\gamma \notin M_{\alpha\gamma}$, that is, $\phi_{\alpha,\alpha\gamma}(r_\alpha)\phi_{\gamma,\alpha\gamma}(s_\gamma) \notin M_{\alpha\gamma}$. But $r_\alpha \cdot R_\alpha \subseteq M_\alpha$ since $r_\alpha \in Q_\alpha$ and $\phi_{\alpha,\alpha\gamma}(R_\alpha) = R_{\alpha\gamma}$ since $\phi_{\alpha,\alpha\gamma}$ is onto. Therefore $\phi_{\alpha,\alpha\gamma}(r_\alpha) \cdot R_{\alpha\gamma} \subseteq M_{\alpha\gamma}$. This is a contradiction to the fact that $\phi_{\alpha,\alpha\gamma}(r_\alpha)\phi_{\gamma,\alpha\gamma}(s_\gamma) \notin M_{\alpha\gamma}$. Hence $P_\alpha = Q_\alpha$ and $P_\alpha = ann\, R_\alpha/M_\alpha$. Thus P_α is primitive. □

The next theorem characterizing primitive ideals of strong Ω-graded rings, where Ω satisfies condition (†) is the main result of this section.

Theorem 2.3. *Let $\mathcal{R}$ be a system of rings and suppose that Ω satisfies condition (†) and that every prime ideal of Ω is completely prime. Let $R = \oplus\mathcal{R}$ be a strong Ω-graded ring. Then there exists a bijective correspondence between the primitive ideals P of R and primitive families $(\Phi, \mathcal{P})$, with Φ a prime ideal of Ω as described in Theorem 1.3.*

Proof. Suppose that P is a primitive ideal of R. Since P is a prime ideal of R, P is of the form $I(\Phi, \mathcal{P})$, that is, P can be written explicitly as $P = I(\Phi) + \cup_{\alpha\in\Omega\setminus\Phi}P_\alpha$, where $P_\alpha \in \mathcal{P}$, for some prime ideal Φ of Ω and a prime family $(\Phi, \mathcal{P})$. Now since P is primitive, $P = ann\, R/M$, where M is a modular maximal left ideal of R. Then M can be expressed as $M = I(\Phi') + \cup_{\alpha\in\Omega\setminus\Phi'}M_\alpha$ for some prime ideal Φ' of Ω and a maximal compatible family $\mathcal{M} = \{M_\alpha\}_{\alpha\in\Omega\setminus\Phi'}$ (see [1]). Since $P \subseteq M$, $\Phi = \Phi'$ and $P_\alpha \subseteq M_\alpha$ for all $\alpha \in \Omega \setminus \Phi$ because $P_\alpha = P \cap R_\alpha \subseteq M \cap R_\alpha = M_\alpha$.

Claim: $(\Phi, \mathcal{P})$ is a primitive family.

Suppose there exists an $r_\alpha \in R_\alpha \setminus P_\alpha$ for some $\alpha \in \Omega \setminus \Phi$ such that for all $\gamma \in \Omega \setminus \Phi$, and all $s_\gamma \in R_\gamma$ we have $r_\alpha \cdot s_\gamma \in M_{\alpha\gamma} \subset M$. This implies that $r_\alpha \cdot \oplus_{\alpha\in\Omega\setminus\Phi}R_\alpha \subseteq M$. Then $r_\alpha \cdot \oplus_{\alpha\in\Omega}R_\alpha \subseteq M$, since $\oplus_{\alpha\in\Phi}R_\alpha \subset M$ by definition of M. Thus

$r_\alpha \in ann\, R/M = P$. Therefore $r_\alpha \in R_\alpha \cap P = P_\alpha$ which is a contradiction to our assumption. Hence for every $r_\alpha \in R_\alpha \setminus P_\alpha$, there exists an $s_\gamma \in R_\gamma$ for some $\gamma \in \Omega \setminus \Phi$ such that $r_\alpha \cdot s_\gamma \notin M_{\alpha\gamma}$. Thus $(\Phi, \mathcal{P})$ is a primitive family.

Conversely, suppose that $(\Phi, \mathcal{P})$, where $\mathcal{P} = (P_\alpha)_{\alpha \in \Omega \setminus \Phi}$, is a primitive family. Then there exists a maximal compatible family of left ideals $\mathcal{M} = (M_\alpha)_{\alpha \in \Omega \setminus \Phi}$ satisfying condition (ii) in Definition 2.1. Let M be the maximal left ideal of R corresponding to $(\Phi, \mathcal{M})$. Then $M = I(\Phi, \mathcal{M})$ (see [1]). Moreover, since $\mathcal{P}$ is a prime family, the ideal $P = I(\Phi, \mathcal{P})$ is a prime ideal of R by Theorem 1.3. Then $P \subseteq M$ because $P_\alpha \subseteq M_\alpha$ for all $\alpha \in \Omega \setminus \Phi$. Therefore $P \cdot r_\alpha \subseteq P \subseteq M$ for all $\alpha \in \Omega$ and $r_\alpha \in R_\alpha$. Thus $P \subseteq ann\, R/M$. We will show that $P = ann\, R/M$. Suppose that $\sum\limits_{\alpha \in \Omega} r_\alpha \in R \setminus P$. Then

$$\begin{aligned} \sum_{\alpha \in \Omega} r_\alpha &= \sum_{\alpha \in \Phi} r_\alpha + \sum_{\alpha \in \Omega \setminus \Phi} r_\alpha \\ &= \sum_{\alpha \in \Phi} r_\alpha + \sum_{\alpha \in \Omega \setminus \Phi} (r_\alpha - \phi_{\alpha,\delta}(r_\alpha)) + \sum_{\alpha \in \Omega \setminus \Phi} \phi_{\alpha,\delta}(r_\alpha), \end{aligned}$$

where δ is the product in a certain order of all $\alpha \in \Omega \setminus \Phi$ in the support of $\sum\limits_{\alpha \in \Omega \setminus \Phi} r_\alpha$. Since $\sum\limits_{\alpha \in \Omega} r_\alpha \notin P$, this implies that $\sum\limits_{\alpha \in \Omega \setminus \Phi} \phi_{\alpha,\delta}(r_\alpha) \notin P$. Hence $\sum\limits_{\alpha \in \Omega \setminus \Phi} \phi_{\alpha,\delta}(r_\alpha) \notin P_\delta$. Then for some $s_\gamma \in R_\gamma, \gamma \in \Omega \setminus \Phi$, $(\sum\limits_{\alpha \in \Omega \setminus \Phi} \phi_{\alpha,\delta}(r_\alpha)) \cdot s_\gamma \notin M_{\delta\gamma}$ by the definition of a primitive family. Therefore $(\sum\limits_{\alpha \in \Omega \setminus \Phi} \phi_{\alpha,\delta}(r_\alpha)) \cdot s_\gamma \notin M$ as $M \cap R_{\delta\gamma} = M_{\delta\gamma}$. This implies that $\sum\limits_{\alpha \in \Omega \setminus \Phi} \phi_{\alpha,\delta}(r_\alpha) \notin ann\, R/M$. Hence $\sum\limits_{\alpha \in \Omega} r_\alpha \notin ann\, R/M$. Since $\sum\limits_{\alpha \in \Omega} r_\alpha$ was arbitrary in $R \setminus P$, it follows that $P = ann\, R/M$. Since $P \neq R$, M is a modular maximal left ideal of R. Hence P is primitive. $\square$

Corollary 2.4. *Let S be a ring, Ω be a semigroup satisfying condition* (†) *such that all prime ideals of Ω are completely prime. Then there exists a bijective correspondence between the primitive ideals P of $S[\Omega]$ and primitive families $(\Phi, \mathcal{P})$, with Φ a prime ideal of Ω, given by $P \longmapsto (\Phi = \{\alpha \in \Omega | S\beta \subseteq P \text{ for all } \beta \in (\alpha)\}, (P'\alpha = P \cap S\alpha)_{\alpha \in \Omega \setminus \Phi})$, where P' is a primitive ideal of S, with inverse given by $(\Phi, \mathcal{P} = (P'\alpha)_{\alpha \in \Omega \setminus \Phi}) \longmapsto I(\Phi, \mathcal{P})$.*

As regular bands are semigroups of idempotents that satisfy condition (†) and since all prime ideals in a regular band are completely prime, the conclusion of Theorem 2.3 applies to strong regular band graded rings. Semilattices are regular bands. Hence the following examples based on semilattices illustrate Theorem 2.3. The prime families in these examples were first considered in [11].

Example 2.5. Let $\Omega = \{\alpha, \beta | \alpha \prec \beta\}$ be a semilattice. Let $R = R_\alpha + R_\beta$ be a strong supplementary semilattice sum of rings over Ω, where $R_\alpha = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$,

$R_\beta = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$, F is a field and the map $\phi_{\alpha,\beta} : R_\alpha \longrightarrow R_\beta$ is the inclusion map. Let $\Phi = \emptyset$, let $P_\alpha = 0$ and $P_\beta = 0$. Then P_α is not a prime ideal of R_α and $P_\alpha = \phi^{-1}_{\alpha,\beta}(P_\beta)$. Since P_β is a prime ideal of R_β, it follows that $(\Phi, \mathcal{P}) = (\emptyset, \{P_\alpha, P_\beta\})$ is a prime family by Corollary 1.5, [1]. Thus condition (i) in Definition is satisfied.

Now let $M_\alpha = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then M_α is a modular maximal left ideal of R_α (in fact, M_α is an ideal). Let $M_\beta = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then M_β is a modular maximal left ideal of R_β. Note that $P_\alpha \subset M_\alpha$ and $P_\beta \subset M_\beta$. Moreover, if $\mathcal{M} = \{M_\alpha, M_\beta\}$, then $(\Phi, \mathcal{M})$ is a maximal compatible family of left ideals as $\phi^{-1}_{\alpha,\beta}(M_\beta) = M_\alpha$. Now since M_α is an ideal, $ann\, R_\alpha/M_\alpha = M_\alpha$, and $ann\, R_\beta/M_\beta = 0 = P_\beta$. Therefore, P_β is primitive but P_α is not. We will show that $I(\Phi, \mathcal{P}) = \{P_\alpha, P_\beta\}$ is a primitive family. Let $r_\alpha \in R_\alpha \setminus P_\alpha$. Then $\phi_{\alpha,\beta}(r_\alpha) \notin P_\beta$. As $P_\beta = ann\, R_\beta/M_\beta$, there exists an $s_\beta \in R_\beta$ such that $\phi_{\alpha,\beta}(r_\alpha) \cdot s_\beta \notin M_\beta$. Now let $r_\beta \in R_\beta \setminus P_\beta$. Then again as $P_\beta = ann\, R_\beta/M_\beta$, there exists an $s_\beta \in R_\beta$ such that $r_\beta \cdot s_\beta \notin M_\beta$. Thus condition (ii) in the definition of a primitive family is satisfied. Hence $(\Phi, \mathcal{P})$ is a primitive family and the corresponding prime ideal $I(\Phi, \mathcal{P})$ is a primitive ideal of R by Theorem 2.3.

Example 2.6. Let $\Omega = \{1, 2, 3, \cdots\}$ be viewed as an upper semilattice with the partial order $\leq$ given by $m \leq n \Leftrightarrow mn = n$. Let F be a field and let M_ω be the ring of infinite matrices $(a_{ij})_{i,j\in\mathbb{N}}$ over F, where $\mathbb{N}$ is the set of natural numbers such that only a finite number of entries a_{ij} are nonzero. Let R_n be the subring of M_ω with $2^n \times 2^n$ upper block triangular matrices, that is, $R_n = \begin{pmatrix} 2^n \times 2^n & 2^n \times 2^n & 0 \\ 0 & 2^n \times 2^n & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $M_\omega = \cup_{n=1}^{\infty} R_n$. Let $R = \oplus_{n\in\Omega} R_n$ be a strong supplementary semilattice sum of rings over Ω with the maps $\phi_{n,m} : R_n \longrightarrow R_m$ being inclusion maps for all $n, m \in Y$ with $n \leq m$. Let $\Phi = \emptyset$. Then Φ is a completely prime ideal of Ω. Let $\mathcal{P} = \{P_n = \langle 0 \rangle \subseteq R_n\}_{n\in\Omega}$. Then clearly,

(i) $P_n = \phi^{-1}_{n,m}(P_m) = \langle 0 \rangle$.

(ii) If $a, b \in R_n$ are such that $aR_m b \subseteq P_m = \langle 0 \rangle$ for all $m \in \Omega$ with $n \leq m$ then $aM_\omega b \subseteq \langle 0 \rangle \subseteq M_\omega$. Therefore either $a = 0 \in P_n$ or $b = 0 \in P_n$, since $\langle 0 \rangle$ is a prime ideal in M_ω. Hence $(\Phi, \mathcal{P})$ is a prime family by [Proposition 1.4, [1]]. Note that P_n is not a prime ideal for any n. Let P be the prime ideal of R corresponding to $(\Phi, \mathcal{P})$. We will show that $(\Phi, \mathcal{P})$ is a primitive family, which would imply that P is also a primitive ideal of R. For each R_n consider the set M_n of $2^n \times 2^n$ upper block triangular matrices such that the first column consists entirely of zeros. Then M_n is a maximal left ideal of R_n for each n and the set $\{M_n | n \in \mathbb{N}\}$ is a maximal compatible family. Note that $P_n \subset M_n$. Let $r_n \in R_n \setminus P_n$. Then $r_n \neq 0$. So r_n has a non-zero entry at some $(i, j)th$ position. Then for any $m > n$, we can choose an element s_m of R_m with a non-zero entry at the $(j, 1)$ position, so that $r_n \cdot s_m = \phi_{n,m}(r_n) \cdot s_m$ has a non-zero entry at the $(i, 1)$ position and hence

$r_n \cdot s_m \notin M_m$. Therefore, $(\Phi, \mathcal{P})$ is a primitive family. Hence P is a primitive ideal of R.

3. Primitive ideals of strong Clifford semigroup graded rings

In this section we shall extend Theorem 2.3 to another class of strong Ω-graded rings, where Ω does not necessarily satisfy condition (†) but has the following properties.

(i) all prime ideals of Ω are completely prime,

(ii) $e\alpha e\beta e = e\alpha\beta e$ for all α, β, e with e idempotent, and

(iii) for every α there exists an idempotent e lying in exactly the same prime ideals as α.

Conditions (i) - (iii) are satisfied by many classes of semigroups such as regular bands, Clifford semigroups, and commutative periodic semigroups. They were considered in the description of prime ideals of semigroup graded rings in [2]. We first characterize primitive ideals of strong Clifford semigroup graded rings and then apply the method to the study of strong Ω-graded rings where Ω satisfies properties (i) - (iii).

Let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$. Let $R = \oplus\mathcal{R}$ be a strong Ω-graded ring, where Ω is a Clifford semigroup. Let E denote the set of idempotents of Ω. Then E is a semilattice and Ω is a strong semilattice of groups, Ω_e, for $e \in E$ (see [7]). Let $R^{(e)} = \oplus_{\alpha\in\Omega_e} R_\alpha$. Then $R^{(e)}$ is a subring of R and as proved in [2], R can be decomposed into a strong supplementary semilattice sum $R = \oplus_{e\in E} R^{(e)}$ via the maps $\phi^{(e,f)} : R^{(e)} \longrightarrow R^{(f)}$ given by $\phi^{(e,f)}(r_\alpha) = \phi_{\alpha,\alpha f}(r_\alpha)$ for $r_\alpha \in R_\alpha$. Moreover, as Ω_e is a group, the maps $\phi_{\alpha,e}$ are isomorphisms for all $\alpha \in \Omega_e$. Hence $R^{(e)}$ is isomorphic to the group ring $R_e[\Omega_e]$. The maps $\phi^{(e,f)}$ can be redefined as $\phi^{(e,f)}(\sum\limits_{\alpha\in\Omega_e} s_\alpha \alpha) = \sum\limits_{\alpha\in\Omega_e} \phi_{e,f}(s_\alpha)\alpha f$, where $s_\alpha \in R_\alpha$ and $e \prec f$. Thus R can be expressed as a strong supplementary semilattice sum of group rings, via the redefined maps $\phi^{(e,f)}$. Then by Theorem 2.3 we have the following result.

Theorem 3.1. *Let Ω be a Clifford semigroup with set of idempotents E and let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$. Suppose that $R = \oplus\mathcal{R}$ is a strong Clifford semigroup graded ring. Then the primitive ideals in R correspond bijectively to primitive families $(\Phi, \mathcal{P})$ with Φ a prime ideal of E and $\mathcal{P}$ a primitive family over the group rings $R_e[\Omega_e]$ with $e \in E \setminus \Phi$ as in Theorem 2.3.*

Corollary 3.2. *Let Ω be a Clifford semigroup with set of idempotents E. Let S be a ring and let $S[\Omega]$ be a semigroup ring. Then the primitive ideals in $S[\Omega]$ correspond bijectively to primitive families $(\Phi, \mathcal{P})$ with Φ a prime ideal of E and $\mathcal{P}$ a primitive family over the group rings $S[\Omega_e]$ with $e \in E \setminus \Phi$.*

Proof. $S[\Omega]$ can be expressed as a strong supplementary semilattice sum, $S[\Omega] =$

$\oplus_{e\in E}S[\Omega_e]$ via the maps $\phi^{(e,f)}(\sum_{\alpha\in\Omega_e} s_\alpha\alpha) = \sum_{\alpha\in\Omega_e} s_\alpha\alpha f$ for $s_\alpha \in S$ and $e \prec f$. Hence the result follows from Theorem 3.2. □

We now apply Theorem 3.2 to strong Ω-graded rings, where Ω satisfies conditions (i) - (iii). As in [2] we shall define a congruence $\equiv$ on Ω that makes $\Omega/\equiv$ into a Clifford semigroup. This would enable us to pass from a strong Ω-graded ring R to the strong $\Omega/\equiv$ graded ring $\overline{R}$ in our description of primitive ideals of R.

We first give an exposition of some important definitions and results stated and proved in [2]. Let η be the least semilattice congruence on Ω. By Proposition 2.5 in [2] for each $e \in E$, $\Gamma_e = \{e\alpha e | \alpha\eta e\}$ is a subgroup of Ω with identity e. Then the congruence $\equiv$ on Ω is defined as follows: $\alpha \equiv \beta \Leftrightarrow$ there is an $e \in E$ such that $\alpha\eta e\eta\beta$ and $e\alpha e = e\beta e$. This definition is independent of the choice of e because then $f\alpha f = f\beta f$ for any $f \in E$ with $f\eta e$. Also if $e, f \in E$ then $e \equiv f$ if and only if $e\eta f$. Moreover, if $\alpha\eta e$ then $\alpha \equiv e\alpha e$. We refer the reader to [2] for details. Let $\phi : \Omega \to \overline{\Omega} = \Omega/\equiv$ be the natural semigroup homomorphism. Let $\tilde{E}$ be a subset of E containing exactly one element e from each η class in E. Then by the above description, $\overline{\Omega}$ can be viewed as the union of the groups Γ_e for $e \in \tilde{E}$. Moreover, $\tilde{E}$ can be made into a semilattice by defining $e \vee f$ to be the unique element of $\tilde{E}$ that is η-equivalent to ef. Hence $\overline{\Omega}$ is a semilattice, $\tilde{E}$, of the groups, Γ_e for $e \in \tilde{E}$. Moreover, by Theorem 2.6, [2], $\overline{\Omega}$ is a Clifford semigroup and $\equiv$ is the least Clifford semigroup congruence on Ω.

Let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$ and let $R = \oplus\mathcal{R}$ be a strong Ω-graded ring. Then $\overline{R} = \oplus_{e\in\tilde{E}, x\in\Gamma_e} R_x$ is a strong Clifford semigroup graded ring and there is a surjective ring homomorphism $\pi : R \longrightarrow \overline{R}$ induced by ϕ. The homomorphism π is given by $\pi(\sum_{\alpha\in\Omega} r_\alpha) = \sum_{e\in\tilde{E}, x\in\Gamma_e} \sum_{\alpha\equiv x} \phi_{\alpha,x}(r_\alpha)$. If K denotes the kernel of π, we have $R/K \cong \overline{R}$. Moreover, by Lemma 3.1, [2], the ideal K is contained in the prime radical of R. Hence every primitive ideal of R contains K as well. Therefore there is a bijective correspondence between the primitive ideals of R and $\overline{R}$. Further since $\phi_{x,e}$ are isomorphisms for all $x \in \Gamma_e$, we have $\overline{R} = \oplus_{e\in\tilde{E}, x\in\Gamma_e} R_x$ is isomorphic to the strong supplementary semilattice sum of group rings $\oplus_{e\in\tilde{E}} R_e[\Gamma_e]$ as in the proof of Theorem 3.2. Hence we have the following result.

Theorem 3.3. *Let Ω be a semigroup having properties* (i), (ii) *and* (iii) *with set of idempotents E and with $\tilde{E}$ as defined above. Let $\mathcal{R}$ be a system of rings over $(\Omega, \prec)$ and let $R = \oplus\mathcal{R}$ be a strong Ω-graded ring. Then the primitive ideals in R correspond bijectively to primitive families $(\Phi, \mathcal{P})$ with Φ a prime ideal of the semilattice $\tilde{E}$ and $\mathcal{P}$ a primitive family over the group rings $R_e[\Gamma_e]$ with $e \in \tilde{E} \setminus \Phi$ as in Theorem 3.2* (1), [2].

Corollary 3.4. *Let Ω be a semigroup having properties* (i), (ii) *and* (iii) *with set of idempotents E and with $\tilde{E}$ as defined above. Let S be a ring and let $S[\Omega]$ be the semigroup ring. Then the primitive ideals in $S[\Omega]$ correspond bijectively to primitive families (Φ, $\mathcal{P}$) with Φ a prime ideal of the semilattice $\tilde{E}$ and $\mathcal{P}$ a primitive family over the group rings $S[\Gamma_e]$ with $e \in \tilde{E} \setminus \Phi$ as in Theorem 3.2* (1), [2].

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