Contemporary Mathematics
Volume **576**, 2012
http://dx.doi.org/10.1090/conm/576/11349

Abelian groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$, Part I

Carol Jacoby, Katrin Leistner, Peter Loth, and Lutz Strüngmann

Dedicated to Professor Rüdiger Göbel in honor of his 70th birthday

Abstract. We consider the class of abelian groups possessing partial decomposition bases in $L^{\delta}_{\infty\omega}$ for δ an ordinal. This class contains the class of Warfield groups which are direct summands of simply presented groups or, alternatively, are abelian groups possessing a nice decomposition basis with simply presented cokernel. We prove a classification theorem using numerical invariants that are deduced from the classical Ulm-Kaplansky and Warfield invariants. This extends earlier work by Barwise-Eklof, Göbel and the authors.

1. Introduction

One aspect of the model theory of groups is to ask for natural generalizations of classical algebraic results in a model theoretic way. Classifications of certain classes of groups (modules) in terms of numerical invariants are an especially good source for such model theoretic descriptions. It was Szmielew [**Sz**] who first took this point of view and considered abelian groups model-theoretically. Later on Barwise and Eklof [**BE**] took up Szmielew's approach and characterized the equivalence classes of torsion abelian groups with respect to the relation of satisfying the same sentences of some infinitary language L, e.g. $L_{\omega_\alpha\omega}$. This results in a model theoretic version of the classification of totally projective p-groups in terms of their Ulm-Kaplansky invariants (see Ulm [**U**], Hill [**H**] and Walker [**Wal**]). In particular Barwise and Eklof obtained as a corollary the classical theorem by Ulm stating that two countable abelian p-groups are isomorphic if and only if their Ulm-Kaplansky invariants coincide. Their proof used derivatives of the Ulm-Kaplansky invariants and a so-called Karp system of isomorphisms I_α that allows the lifting of partial mappings to global ones. Recall that the usual axioms for abelian groups can be stated in the lower predicate calculus (i.e. in $L_{\omega\omega}$), however it is necessary to allow languages with infinite expressions to characterize torsion groups or simple groups. For instance, the compactness theorem shows that abelian torsion groups are not axiomatizable in $L_{\omega\omega}$.

2010 *Mathematics Subject Classification.* Primary 03C52, 03C60, 13C05; Secondary 03E10, 20K21, 20K25, 20K35.

The authors would like to thank the German Israeli Foundation for their support project No. I-963-98.6/2007; and the German Research Foundation for their support project STR 627/1-6 as well as the PPP-program of DAAD-AVČR.

Extending the classification of totally projective p-groups algebraically it was Warfield [**War2**] who introduced new invariants, the so-called Warfield invariants, to classify up to isomorphism p-local abelian groups possessing a nice decomposition basis with simply presented cokernel. Motivated by the result of Barwise and Eklof the first author [**J1**], [**J2**] considered Warfield's theorem and proved an analogous result in $L_{\infty\omega}$ for a larger class of R-modules in the case of $R = \mathbb{Z}_p$ and also in the global case of $R = \mathbb{Z}$. Again variants of the classical Warfield invariants were needed. Independently the second, third and fourth authors proved similar results together with Göbel (see [**GLLS**]).

Passing to $L^{\delta}_{\infty\omega}$ the authors were able to prove in the local case analogous results using Karp systems that consist entirely of partial isomorphisms that preserve heights up to ordinals $\leq \omega\delta$. In this paper we will show the global model theoretic classification of modules with partial decomposition bases in $L^{\delta}_{\infty\omega}$.

The rest of the paper is organized as follows: In section 2 we recall the basic definitions and notation concerning p-groups and Warfield modules from module theory. Then in section 3 we review the basics of model theory needed for our paper. Section 4 contains the known classifications from [**BE**], [**GLLS**], [**J1**] and [**J2**] in the language $L_{\infty\omega}$. Section 5 contains a classification in $L^{\delta}_{\infty\omega}$ for $\mathbb{Z}_p$-modules with partial decomposition bases which can be found in [**JL**]. Finally, section 6 contains our main theorem which is the classification theorem in $L^{\delta}_{\infty\omega}$ in the global case for groups with partial decomposition bases.

For notation and terminology on abelian groups and on model theory, the reader may refer to [**F1**], [**F2**], [**L**] and [**R**].

2. Notations and terminology

Throughout this paper, R will denote an arbitrary principal ideal domain, unless stated otherwise, and G will denote an R-module. A module over $\mathbb{Z}$ is referred to as a *group*. The torsion part of G is denoted by tG. For each ordinal α and prime $p \in R$, a submodule $p^{\alpha}G$ is defined as follows: $pG = \{pg : g \in G\}$, $p^{\alpha+1}G = p(p^{\alpha}G)$, and $p^{\alpha}G = \bigcap_{\beta<\alpha} p^{\beta}G$ if α is a limit ordinal. The *p-length* of G is the smallest ordinal τ such that $p^{\tau}G = p^{\tau+1}G$. With every $x \in G$, we associate its *p-height* $|x|_p$, that is, $|x|_p = \alpha$ if $x \in p^{\alpha}G \setminus p^{\alpha+1}G$ and $|x|_p = \infty$ if $x \in p^{\infty}G = \bigcap_{\alpha} p^{\alpha}G$. If S is a subset of G, then $\langle S\rangle$ denotes the submodule of G generated by S, and $\langle S\rangle^0$ denotes the set of all elements $x \in G$ such that $rx \in \langle S\rangle$ for some $0 \neq r \in R$. If x is an element of S, then the p-height of x computed in G coincides with the p-height of x computed in $\langle S\rangle^0$.

For a prime p, $\mathbb{Z}_p$ denotes the ring of integers localized at p. Notice that the natural map $G \to G_p = G \otimes \mathbb{Z}_p$ with $x \mapsto x_p = x \otimes 1$ preserves p-heights (cf. [**F2**, Part 2, Lemma 13]). Suppose S and T are submodules of R-modules G and H, respectively. For an ordinal α, an isomorphism $f : S \to T$ is called *α-height-preserving* if for all $x \in S$ and primes $p \in R$ we have $\min\{|x|_p, \alpha\} = \min\{|f(x)|_p, \alpha\}$ where all p-heights are computed in G and H, respectively. Let $G[p] = \{x \in G : px = 0\}$. Recall that the *Ulm-Kaplansky invariants* of G are defined by

$$u_p(\alpha, G) = \dim(p^{\alpha}G)[p]/(p^{\alpha+1}G)[p]$$

for α an ordinal, and

$$u_p(\infty, G) = \dim(p^{\infty}G)[p].$$

The subscript p may be dropped if it is clear which prime p is considered. Ulm [**U**] proved that these cardinal numbers are isomorphism invariants of countable torsion groups.

We adopt the convention that $\infty > \alpha$ for α an ordinal or ∞. An *Ulm sequence* is an increasing sequence $\overline{\beta} = (\beta_i : i < \omega)$ where each β_i is an ordinal or the symbol ∞. Two Ulm sequences $\overline{\beta} = (\beta_i)$ and $\overline{\gamma} = (\gamma_i)$ are called *equivalent*, and we write $\overline{\beta} \sim \overline{\gamma}$, if there exist $k, l < \omega$ such that $\beta_{i+k} = \gamma_{i+l}$ for all $i < \omega$. The equivalence class of $\overline{\beta}$ is denoted by $[\overline{\beta}]$. The *p-Ulm sequence of $x \in G$* is the sequence $U_p(x) = (|p^i x|_p : i < \omega)$.

For an element x of a group G the *Ulm matrix of x* is the doubly infinite $\mathbf{P} \times \omega$ matrix $U(x)$ having $U_p(x)$ as its p-th row. More generally, an *Ulm matrix* is a matrix $A = [a_{p,i}]_{(p,i) \in \mathbf{P} \times \omega}$ such that each row is an Ulm sequence. If n is a positive integer, we let nA be the Ulm matrix having $a_{p,i+|n|_p}$ as its (p,i) entry where $|n|_p$ is the p-height of n in $\mathbb{Z}$. It follows that $nU(x) = U(nx)$. If $A = [a_{p,i}]$ and $B = [b_{p,i}]$, we write $A \geq B$ in case $a_{p,i} \geq b_{p,i}$ for all (p,i). Two Ulm matrices A and B are called *compatible* if there are positive integers m and n such that $mB \geq A$ and $nA \geq B$. This yields an equivalence relation, and the equivalence classes are called *compatibility classes*. Note that if two Ulm matrices are compatible, their p-th rows coincide for almost all primes p. A subset X of G is called a *decomposition set* if

(1) all elements of X are independent and have infinite order;
(2) for each $x = k_1x_1 + \ldots + k_nx_n$ ($k_1, \ldots, k_n \in R$, $x_1, \ldots, x_n \in X$) and prime $p \in R$, we have $|x|_p = \min\{|k_i x_i|_p\}$.

X is called a *decomposition basis for G* if in addition, $G/\langle X\rangle$ is torsion. A decomposition basis X for G is called *nice* if for every prime p, $\langle X\rangle_p$ is a p-nice subgroup of G_p, that is, if every coset $x + \langle X\rangle_p$ ($x \in G_p$) has an element of maximal p-height. Groups ($\mathbb{Z}_p$-modules, resp.) possessing a nice decomposition basis X with simply presented quotient $G/\langle X\rangle$ are called *Warfield groups* (*Warfield modules*, resp.).

Stanton defined cardinal numbers that, together with the Ulm-Kaplansky invariants, form a complete set of isomorphism invariants for Warfield groups (see Hunter-Richman [**HR**], Stanton [**St**]).

In [**J2**], Jacoby generalized the concept of decomposition basis: If G is an R-module, a system $\mathcal{C}$ is called a *partial decomposition basis for G* if

(1) $\mathcal{C}$ is a nonempty collection of finite subsets of G;
(2) if $X \in \mathcal{C}$, then X is a decomposition set;
(3) if $X \in \mathcal{C}$ and $x \in G$, there is $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $x \in \langle Y\rangle^0$.

It is clear that if X is a decomposition basis for G, then the collection of all finite subsets of X is a partial decomposition basis for G.

3. Model-theoretic preliminaries

Throughout this paper, we will consider the language $L_{\infty\omega}$ which is an extension of an ordinary first order language L that allows infinite conjunctions and disjunctions and has a variable v_α for every ordinal α. In this paper we will take L to be the language of group theory with 0, + and $-$.

We define for each ordinal α a collection $L^\alpha_{\infty\omega}$ of formulas as the smallest collection F of formulas which contains the atomic formulas and is closed under the following logical operations:

(L1) If $\varphi \in F$, then $\neg\varphi \in F$.
(L2) If $\Phi \subseteq F$, then $\bigwedge \Phi, \bigvee \Phi \in F$.
(L3) If $\varphi \in L^{\beta}_{\infty\omega}$ for $\beta < \alpha$ and υ is a variable, then $\exists \upsilon\, \varphi, \forall \upsilon\, \varphi \in F$.

In (L2) $\bigwedge \Phi$ ($\bigvee \Phi$, resp.) denotes the conjunction (disjunction, resp.) of all elements of the set Φ. Then $L_{\infty\omega} = \bigcup_{\alpha} L^{\alpha}_{\infty\omega}$. The *quantifier rank* $qr(\varphi)$ of a formula $\varphi \in L_{\infty\omega}$ is defined to be the least ordinal α such that $\varphi \in L^{\alpha}_{\infty\omega}$.

Models for a language are understood to be sets of elements, which, in combination with the language-specific constants, functions and relations, satisfy the axioms of the language and in which every possible sentence has a distinct truth value. Models are denoted by $\mathfrak{A} = \langle A, ... \rangle$, where A is a set of elements called the universe. If $\varphi \in L_{\infty\omega}$ is a formula with at most n variables, $a_1, ..., a_n \in A$ and $\varphi(a_1, ..., a_n)$ is true, we write $\mathfrak{A} \models \varphi[a_1, ..., a_n]$, and accordingly for a sentence φ which is true, $\mathfrak{A} \models \varphi$.

Let α be an ordinal. Then two models $\mathfrak{A} = \langle A, ... \rangle$ and $\mathfrak{B} = \langle B, ... \rangle$ for $L_{\infty\omega}$ are called $L^{\alpha}_{\infty\omega}$-*equivalent* (resp. $L_{\infty\omega}$-equivalent), and we write $\mathfrak{A} \equiv_{\alpha} \mathfrak{B}$ (resp. $\mathfrak{A} \equiv_{\infty} \mathfrak{B}$), if for all sentences $\varphi \in L^{\alpha}_{\infty\omega}$ (resp. $\varphi \in L_{\infty\omega}$) we have

$$\mathfrak{A} \models \varphi \text{ if and only if } \mathfrak{B} \models \varphi.$$

$L^{\alpha}_{\infty\omega}$-equivalent and $L_{\infty\omega}$-equivalent models can be characterized using partial isomorphisms between them:

THEOREM 3.1 (Karp [**K**]). *Let $\mathfrak{A} = \langle A, \dots \rangle$ and $\mathfrak{B} = \langle B, \dots \rangle$ be models for $L_{\infty\omega}$ and δ an ordinal or the symbol ∞. Then the following are equivalent:*

(1) $\mathfrak{A} \equiv_{\delta} \mathfrak{B}$*;*
(2) *For each ordinal $\nu \leq \delta$ there is a non-empty set I_{ν} of isomorphisms on finitely generated substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that*
   (a) *if $\nu \leq \mu$, then $I_{\mu} \subseteq I_{\nu}$;*
   (b) *if $\nu < \delta$, $f \in I_{\nu+1}$ and $x \in A$ ($y \in B$, resp.), then f extends to a map $f' \in I_{\nu}$ such that $x \in domain(f')$ ($y \in range(f')$, resp.).*

4. Equivalence in $L_{\infty\omega}$

Ulm [**U**] gave a complete classification of countable p-groups in terms of the Ulm-Kaplansky invariants defined earlier. Warfield [**War2**] extended this theorem to Warfield modules by introducing invariants for an R-module M with decomposition basis X, where R is a discrete valuation ring with prime p. In the definitions that follow e will be an equivalence class of Ulm sequences, c a compatibility class of Ulm matrices and p a prime. Let

$$w(e, M) = |\{x \in X : U_p(x) \in e\}|.$$

Stanton [**St**] extended these invariants to groups and proved that they classify Warfield groups up to isomorphism. If G has a decomposition basis X, the *Warfield invariants* are defined by

$$w(c, p, e, G) = |\{x \in X : U(x) \in c \text{ and } U_p(x) \in e\}|.$$

Ulm's Theorem was extended by Barwise and Eklof [**BE**] who modified the Ulm-Kaplansky invariants: for a p-group G and any ordinal α, define $\hat{u}(\alpha, G) = \min\{u(\alpha, G), \omega\}$ and $\hat{u}(\infty, G) = \min\{u(\infty, G), \omega\}$. They proved that these invariants classify all torsion groups in $L_{\infty\omega}$.

Let M be an R-module with partial decomposition basis $\mathcal{C}$ where R is a discrete valuation ring with prime p. Jacoby [**J2**] defined $\hat{w}(e, M)$ to be the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $U_p(x_i) \in e$ for all $i = 1, \ldots, n$. If no such n exists, put $\hat{w}(e, M) = \omega$. These invariants, along with the invariants of Barwise and Eklof, classify all R-modules with partial decomposition bases up to $L_{\infty\omega}$-equivalence.

Similarly for groups, Jacoby adapted $w(c, p, e, G)$ as follows: Let $\hat{w}(c, p, e, G)$ be the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $U(x_i) \in c$ and $U_p(x_i) \in e$ for all $i = 1, \ldots, n$. If no such n exists, put $\hat{w}(c, p, e, G) = \omega$. The next result, which will be needed, shows that this ordinal is independent of the choice of partial decomposition basis. First we need a lemma.

Lemma 4.1 (Jacoby [**J2**]). *Let M be a module over a principal ideal domain R which has partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$. Let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then there are decomposition sets X' and Y' such that $X \subseteq X', Y \subseteq Y', X'$ and Y' are unions of ascending chains of elements of $\mathcal{C}$ and $\mathcal{D}$ respectively, and $\langle X' \rangle^0 = \langle Y' \rangle^0$.*

Proof. We will define, by induction on i, sets $X_i \in \mathcal{C}$, $Y_i \in \mathcal{D}$. Let $X_0 = X$ and $Y_0 = Y$. Suppose X_i and Y_i have been chosen. Choose $X_{i+1} \in \mathcal{C}$ such that $X_i \subseteq X_{i+1}$ and $Y_i \subseteq \langle X_{i+1} \rangle^0$, by a finite number of applications of condition (3) of the definition. Then choose similarly $Y_{i+1} \in \mathcal{D}$ such that $Y_i \subseteq Y_{i+1}$ and $X_{i+1} \subseteq \langle Y_{i+1} \rangle^0$.

Let $X' = \bigcup_{i \in \omega} X_i$ and $Y' = \bigcup_{i \in \omega} Y_i$. We claim that $\langle X' \rangle^0 \subseteq \langle Y' \rangle^0$. Let $z \in \langle X' \rangle^0$. Then for some $n \in \mathbb{Z}$, $a \in R$ and $x_1, ..., x_n \in X'$, $az \in \langle x_1, \ldots, x_n \rangle$. Choose $X_i \supseteq \{x_1, ..., x_n\}$. Then $\{x_1, ..., x_n\} \subseteq \langle Y_i \rangle^0$, so $\{bx_1, ..., bx_n\} \subseteq \langle Y_i \rangle$ for some $b \in R$. But then $baz \in \langle Y_i \rangle \subseteq \langle Y' \rangle$, so $z \in \langle Y' \rangle^0$. Similarly, $\langle Y' \rangle^0 \subseteq \langle X' \rangle^0$. $\square$

Theorem 4.2 (Jacoby [**J2**]). *Suppose G is a group with partial decomposition basis $\mathcal{C}$ and assume c is a compatibility class of Ulm matrices, p is a prime and e is an equivalence class of Ulm sequences such that $\hat{w}(c, p, e, G) \geq n$. If $\mathcal{D}$ is any partial decomposition basis for G and $Y \in \mathcal{D}$, then there exists $\tilde{Y} \in \mathcal{D}$ such that $Y \subseteq \tilde{Y}$ and $\tilde{Y}$ contains elements $y_1, \ldots, y_n$ such that $U(y_i) \in c$ and $U_p(y_i) \in e$ for all $i = 1, \ldots, n$.*

Proof. Let c, p and e be given and suppose $\hat{w}(c, p, e, G) \geq n$. Then by definition there is a $X \in \mathcal{C}$ containing elements $x_1, \ldots, x_n$ such that $U(x_i) \in c$ and $U_p(x_i) \in e$ for $1 \leq i \leq n$. Let Y be as given and choose X' and Y' as in Lemma 4.1. Then X' and Y' are both decomposition bases for $\langle X' \rangle^0 = \langle Y' \rangle^0$. Stanton [**St**] proved that $w(c, p, e, G)$ is independent of the choice of decomposition basis, so $w(c, p, e, \langle Y' \rangle^0) \geq n$. Thus Y' contains elements $y_1, \ldots, y_n$ such that $U(y_i) \in c$ and $U_p(y_i) \in e$ for $1 \leq i \leq n$. Choose $\tilde{Y} \in \mathcal{D}$ containing Y and $y_1, \ldots, y_n$. $\square$

Since the natural map $G \to G_p$ preserves p-heights, repeated application of this theorem yields

Corollary 4.3. *Suppose G is a group with partial decomposition basis, p is a prime and e is an equivalence class of Ulm sequences. Then $\hat{w}(e, G_p) = \min\{\sum_c \hat{w}(c, p, e, G), \omega\}$.*

5. Equivalence in $L^{\delta}_{\infty\omega}$: Local classification

In [**BE**], Barwise and Eklof classified p-groups up to $L^{\delta}_{\infty\omega}$-equivalence:

THEOREM 5.1 (Barwise-Eklof [**BE**]). *Let G and H be p-groups and let δ be an ordinal. Suppose*

(1) $\hat{u}(\alpha, G) = \hat{u}(\alpha, H)$ *for all ordinals* $\alpha < \omega\delta$;
(2) *if* $length(G) < \omega\delta$, *then* $\hat{u}(\infty, G) = \hat{u}(\infty, H)$.

Then $G \equiv_\delta H$. *If δ is a limit ordinal, the converse also holds.*

Let α be an ordinal. Two Ulm sequences (β_i) and (γ_i) are said to be *equal up to* α, and we write $(\beta_i) =_\alpha (\gamma_i)$, if

$$\min\{\beta_i, \alpha\} = \min\{\gamma_i, \alpha\}$$

for all $i < \omega$. Two equivalence classes e and e' of Ulm sequences are called *α-equivalent*, and we write $e \sim_\alpha e'$, if there are Ulm sequences $(\beta_i) \in e$ and $(\gamma_i) \in e'$ which are equal up to α. In this case, we will also call any two Ulm sequences $u \in e$ and $u' \in e'$ *α-equivalent* and write $u \sim_\alpha u'$. It is clear that $u \sim_\alpha u'$ if and only if $[u] \sim_\alpha [u']$. If R is a discrete valuation ring, define

$$\hat{w}_\alpha(e, M) = \min\{\sum_{e' \sim_\alpha e} \hat{w}(e', M), \omega\}.$$

The next lemma will be useful and can be easily verified.

LEMMA 5.2. *Let M be a $\mathbb{Z}_p$-module with partial decomposition basis. Then M has a partial decomposition basis $\mathcal{C}$ such that*

(1) $X \in \mathcal{C}, X' \subseteq X \Rightarrow X' \in \mathcal{C}$;
(2) $\{x_1, \ldots, x_n\} \in \mathcal{C}, a_1, \ldots, a_n \in \mathbb{Z}_p \backslash \{0\} \Rightarrow \{a_1x_1, \ldots, a_nx_n\} \in \mathcal{C}$.

The following classification of $\mathbb{Z}_p$-modules with partial decomposition bases in $L^\delta_{\infty\omega}$ is the main result of [**JL**] and will be generalized in Theorem 6.10:

THEOREM 5.3 (Jacoby-Loth [**JL**]). *Let M and N be $\mathbb{Z}_p$-modules with partial decomposition bases and let δ be an ordinal. Suppose*

(1) $\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ *for all* $\alpha < \omega\delta$;
(2) $\hat{w}_{\omega(\nu+1)}(e, M) = \hat{w}_{\omega(\nu+1)}(e, N)$ *for all equivalence classes e of Ulm sequences and ordinals* $\nu < \delta$;
(3) *if* $length(tM) < \omega\delta$, *then* $\hat{u}(\infty, M) = \hat{u}(\infty, N)$.

Then $M \equiv_\delta N$.

PROOF. A detailed proof can be found in [**JL**]. We would like to give the basic idea of this proof as it uses a construction which is needed for our proof of Theorem 6.10. Suppose M and N are $\mathbb{Z}_p$-modules satisfying conditions (1)-(3) of Theorem 5.3 and let $\mathcal{C}_M$ and $\mathcal{C}_N$ be partial decomposition bases for M and N, respectively, as in Lemma 5.2. Consider the system $\{I_\nu : \nu \leq \delta\}$ where each I_ν is the set of all maps $f : S \to T$ such that there are decomposition sets $X \in \mathcal{C}_M$ and $Y \in \mathcal{C}_N$ with $f(X) = Y$ satisfying the following:

(i) S and T are finitely generated submodules of M and N, respectively;
(ii) f is an $\omega\nu$-height-preserving isomorphism;
(iii) $X \subseteq S \subseteq \langle X \rangle^0$ and $Y \subseteq T \subseteq \langle Y \rangle^0$.

It suffices to show that the system $\{I_\nu : \nu \leq \delta\}$ satisfies condition (2) of Karp's Theorem 3.1. Clearly, each set I_ν is non-empty as it contains the zero function. Let $f : S \to T$ be a map in $I_{\nu+1}$ ($\nu < \delta$) with associated decomposition sets $X \in \mathcal{C}_M$ and $Y \in \mathcal{C}_N$, and let $a \in M$. Then there is $X' = X \cup \{x_1, \ldots, x_m\} \in \mathcal{C}_M$ such that

$a \in \langle X' \rangle^0$. Since M and N have identical modified Warfield invariants (condition (2) of Theorem 5.3), a set $Y' = Y \cup \{y_1, \ldots, y_m\} \in \mathcal{C}_N$ can be constructed such that f extends to a map

$$f' : \langle S, x_1', \ldots, x_m' \rangle \to \langle T, y_1', \ldots, y_m' \rangle$$

in $I_{\nu+1}$ for some nonzero multiples x_i' of x_i and y_i' of y_i, where x_i' is mapped onto y_i' $(i = 1, \ldots, m)$. Finally, f' extends to a map $g \in I_\nu$ such that $a \in domain(g)$ due to the imposed conditions on the modified Ulm-Kaplansky invariants (conditions (1) and (3) of Theorem 5.3). By symmetry, condition (2) of Theorem 3.1 is satisfied and $M \equiv_\delta N$ follows. □

Remark 5.4. *It is easy to see that condition (2) in Theorem 5.3 follows from*
(2') $\hat{w}_{\omega\delta}(e, M) = \hat{w}_{\omega\delta}(e, N)$ *for all equivalence classes e of Ulm sequences.*

6. Equivalence in $L^\delta_{\infty\omega}$: Global classification

Let $A = [a_{p,i}]$ and $B = [b_{p,i}]$ be Ulm matrices, and let α be an ordinal. We say that A and B *are equal up to* α and write $A =_\alpha B$ if

$$\min\{a_{p,i}, \alpha\} = \min\{b_{p,i}, \alpha\}$$

for all primes p and $i < \omega$. We then define a relation on compatibility classes of Ulm matrices as follows: $c \sim_\alpha c'$ if and only if there are $A \in c$ and $B \in c'$ such that $A =_\alpha B$. This can be easily verified to be an equivalence relation. If $c \sim_\alpha c'$, then we call any Ulm matrices $C \in c$ and $C' \in c'$ *α-compatible* and write $C \sim_\alpha C'$. Notice that if any two Ulm matrices are α-compatible, their respective p-rows are equal up to α for almost all primes p. Suppose G and H are groups with elements $x \in G$ and $y \in H$ having α-compatible Ulm matrices such that $U_p(x) \sim_\alpha U_p(y)$ for all primes p. Clearly, then there are positive integers k and l such that $U_p(kx) =_\alpha U_p(ly)$ for all primes p.

For any group G with partial decomposition basis, ordinal α, compatibility class c of Ulm matrices, prime p and equivalence class e of Ulm sequences we define

$$\hat{w}_\alpha(c, p, e, G) = \min\{\sum_{c' \sim_\alpha c, e' \sim_\alpha e} \hat{w}(c', p, e', G), \omega\}.$$

Notice that for a finite decomposition set X we have

$$\begin{aligned}\hat{w}_\alpha(c, p, e, \langle X \rangle^0) &= \sum_{c' \sim_\alpha c, e' \sim_\alpha e} |\{x \in X : U(x) \in c' \text{ and } U_p(x) \in e'\}| \\ &= |\{x \in X : [U(x)] \sim_\alpha c \text{ and } [U_p(x)] \sim_\alpha e\}|.\end{aligned}$$

The following fact will be needed and is easily verified:

Lemma 6.1. *Suppose G is a group with partial decomposition basis $\mathcal{C}$. If $X \in \mathcal{C}$, then $\hat{w}_\alpha(c, p, e, G) \geq \hat{w}_\alpha(c, p, e, \langle X \rangle^0)$ for any ordinal α, compatibility class c of Ulm matrices, prime p and equivalence class e of Ulm sequences.*

The next result is the analog to Theorem 4.2 and shows that the cardinals $\hat{w}_\alpha(c, p, e, G)$ are independent of the choice of partial decomposition basis:

Theorem 6.2. *Let G be a group with partial decomposition basis $\mathcal{C}$, α an ordinal and n a positive integer. If $\hat{w}_\alpha(c, p, e, G) \geq n$ and $Y \in \mathcal{C}$, then there exists $Y' \in \mathcal{C}$ such that $Y \subseteq Y'$ and Y' contains elements $y_1, \ldots, y_n$ such that $[U(y_i)] \sim_\alpha c$ and $[U_p(y_i)] \sim_\alpha e$ for all $i = 1, \ldots, n$.*

Proof. Letting $E = \{(c', e') : \hat{w}(c', p, e', G) \neq 0\}$, we have

$$\hat{w}_\alpha(c, p, e, G) = \min\{\sum_{c' \sim_\alpha c, e' \sim_\alpha e, (c', e') \in E} \hat{w}(c', p, e', G), \omega\}.$$

Let $(c', e') \in E$ and suppose $\hat{w}(c', p, e', G) \geq k$. By Theorem 4.2 there is $Y' \in \mathcal{C}$ such that $Y \subseteq Y'$ and Y' has k elements x with $U(x) \in c' \sim_\alpha c$ and $U_p(x) \in e' \sim_\alpha e$. Repeat this for all elements in E until at least n such elements have been adjoined. □

Corollary 6.3. *Let G be a group with partial decomposition basis $\mathcal{C}$, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Then $\hat{w}_\alpha(c, p, e, G)$ is the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $[U(x_i)] \sim_\alpha c$ and $[U_p(x_i)] \sim_\alpha e$ for all $i = 1, \ldots, n$. If no such n exists, $\hat{w}_\alpha(c, p, e, G) = \omega$.*

Proof. Suppose there is a largest integer n such that there is $X \in \mathcal{C}$ containing n elements x satisfying $[U(x)] \sim_\alpha c$ and $[U_p(x)] \sim_\alpha e$. Then by Theorem 6.2, $\hat{w}_\alpha(c, p, e, G) \leq n$. On the other hand, $\hat{w}_\alpha(c, p, e, G) \geq \hat{w}_\alpha(c, p, e, \langle X \rangle^0) = n$ by Lemma 6.1. If no such n exists, $\hat{w}_\alpha(c, p, e, G) = \omega$ by Lemma 6.1. □

Corollary 6.4. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$, respectively, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Suppose $\hat{w}_\alpha(c, p, e, G) = \hat{w}_\alpha(c, p, e, H)$, $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and*

$$\hat{w}_\alpha(c, p, e, \langle X \rangle^0) > \hat{w}_\alpha(c, p, e, \langle Y \rangle^0).$$

Then there exists $Y' \in \mathcal{D}$ such that $Y \subseteq Y'$ and there is $y \in Y' \backslash Y$ such that $[U(y)] \sim_\alpha c$ and $[U_p(y)] \sim_\alpha e$.

Proof. Let $n = \hat{w}_\alpha(c, p, e, \langle Y \rangle^0)$. Then

$$\hat{w}_\alpha(c, p, e, H) = \hat{w}_\alpha(c, p, e, G) \geq \hat{w}_\alpha(c, p, e, \langle X \rangle^0) > \hat{w}_\alpha(c, p, e, \langle Y \rangle^0) = n.$$

By Theorem 6.2, there is $Y' \in \mathcal{D}$ such that $Y \subseteq Y'$ and Y' contains $n+1$ elements y such that $[U(y)] \sim_\alpha c$ and $[U_p(y)] \sim_\alpha e$. Since Y contains only n such elements, one of them is in $Y' \backslash Y$. □

The following lemma will be useful:

Lemma 6.5 (Stanton [**St**]). *Let G be a group with decomposition basis X, p a prime and $x_1, x_2 \in X$ with α-compatible Ulm matrices. Then there are elements $y_1, y_2 \in \langle X \rangle$ such that $U_p(y_1) =_\alpha U_p(x_2)$, $U_p(y_2) =_\alpha U_p(x_1)$ and $U_q(y_1) =_\alpha U_q(x_1)$, $U_q(y_2) =_\alpha U_q(x_2)$ for all primes $q \neq p$. The set $Y = (X \backslash \{x_1, x_2\}) \cup \{y_1, y_2\}$ is a decomposition basis for G and $\langle X \rangle = \langle Y \rangle$.*

Stanton proved this without the "up to α" conditions. The proof applies just as well in this case. Notice that in the lemma above, the elements x_1 and y_1 have α-compatible Ulm matrices.

Lemma 6.6 (Jacoby [**J2**]). *Suppose G is a group with partial decomposition basis. Then G has a partial decomposition basis $\mathcal{C}$ for G such that*

(1) $X \in \mathcal{C}, X' \subseteq X \Rightarrow X' \in \mathcal{C}$;
(2) $X \in \mathcal{C}, \langle X \rangle = \langle Y \rangle$, Y *finite decomposition set* $\Rightarrow Y \in \mathcal{C}$;
(3) $\{x_1, \ldots, x_n\} \in \mathcal{C}, a_1, \ldots, a_n \in \mathbb{Z} \backslash \{0\} \Rightarrow \{a_1x_1, \ldots, a_nx_n\} \in \mathcal{C}$.

PROOF. The lemma can be easily verified by taking the union of a chain of partial decomposition bases for G that alternately satisfy conditions (1) and (3) and condition (2). $\square$

LEMMA 6.7. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$ satisfying conditions (1) and (2) of Lemma 6.6. Suppose α is an ordinal such that $\hat{w}_\alpha(c,p,e,G) = \hat{w}_\alpha(c,p,e,H)$ for every compatibility class c of Ulm matrices, prime p and equivalence class e of Ulm sequences. Assume $X \cup \{x\} \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that*

$$\hat{w}_\alpha(c,p,e,\langle X\rangle^0) = \hat{w}_\alpha(c,p,e,\langle Y\rangle^0)$$

for all c, p and e. Then there exists an element $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$ and

$$\hat{w}_\alpha(c,p,e,\langle X \cup \{x\}\rangle^0) = \hat{w}_\alpha(c,p,e,\langle Y \cup \{y\}\rangle^0)$$

for all c, p and e. In fact, $U(x) \sim_\alpha U(y)$ and $U_p(x) \sim_\alpha U_p(y)$ for all primes p.

PROOF. Suppose $x \in G\backslash X$ and let c_0 be the compatibility class containing $U(x)$, p_0 a prime and e_0 the equivalence class of Ulm sequences containing $U_{p_0}(x)$. Then

$$\hat{w}_\alpha(c_0,p_0,e_0,\langle X \cup \{x\}\rangle^0) = \hat{w}_\alpha(c_0,p_0,e_0,\langle X\rangle^0) + 1 > \hat{w}_\alpha(c_0,p_0,e_0,\langle Y\rangle^0).$$

By Corollary 6.4 and condition (1) of Lemma 6.6 there is an element $z \in H\backslash Y$ satisfying

$$(*) \quad Y \cup \{z\} \in \mathcal{D},\ U(z) \sim_\alpha c_0 \text{ and } [U_{p_0}(z)] \sim_\alpha e_0.$$

Then $U(x)$ and $U(z)$ are α-compatible and $U_p(x)$ and $U_p(z)$ are α-equivalent for all but finitely many primes p, say, $p_1, \ldots, p_n$. We will show by induction on n that z can be replaced by an element $y \in H$ satisfying $(*)$ such that $U_p(x)$ and $U_p(y)$ are α-equivalent for all primes p. For $n = 0$ there is nothing to show, so assume the assertion is true for $n-1$ and let $e_n = [U_{p_n}(x)]$. By our assumption, $[U_{p_n}(z)] \not\sim_\alpha e_n$ and therefore

$$\hat{w}_\alpha(c_0,p_n,e_n,\langle Y \cup \{z\}\rangle^0) = \hat{w}_\alpha(c_0,p_n,e_n,\langle Y\rangle^0) < \hat{w}_\alpha(c_0,p_n,e_n,\langle X \cup \{x\}\rangle^0).$$

By Corollary 6.4, there is $z' \in H$ such that $Y \cup \{z,z'\} \in \mathcal{D}$, $[U(z')] \sim_\alpha c_0$ and $[U_{p_n}(z')] \sim_\alpha e_n$. Now we apply Lemma 6.5 to the group $M = \langle Y \cup \{z,z'\}\rangle^0$ with decomposition basis $Y \cup \{z,z'\}$, elements z, z' and prime p_n. Then there are elements $y, y' \in \langle Y \cup \{z,z'\}\rangle$ such that $Y \cup \{y,y'\}$ is a decomposition basis for M, $\langle Y,y,y'\rangle = \langle Y,z,z'\rangle$ and $U_{p_n}(y) =_\alpha U_{p_n}(z')$, $U_{p_n}(y') =_\alpha U_{p_n}(z)$ and $U_q(y) =_\alpha U_q(z)$, $U_q(y') =_\alpha U_q(z')$ whenever $q \neq p_n$. Notice that condition $(*)$ holds for y as $Y \cup \{y\} \in \mathcal{D}$ by conditions (1) and (2) of Lemma 6.6, y and z have α-compatible Ulm matrices and $U_{p_0}(y) = U_{p_0}(z)$ up to α. To complete the induction, we need to verify that $U_p(x)$ and $U_p(y)$ are α-equivalent for all but $n-1$ primes p. Indeed, $U_q(y) =_\alpha U_q(z) \sim_\alpha U_q(x)$ whenever $q \notin \{p_1, \ldots, p_n\}$, and $U_{p_n}(y) =_\alpha U_{p_n}(z') \sim_\alpha U_{p_n}(x)$. This completes the proof. $\square$

LEMMA 6.8 (Jacoby [**J2**]). *Let G be a group with decomposition basis X and S a finitely generated subgroup of G such that $S \cap \langle X\rangle = \langle S \cap X\rangle$. If $y \in X$ $(y \notin S)$, then there is a positive integer n satisfying $|mny + s|_p = \min\{|mny|_p, |s|_p\}$ for all $m \in \mathbb{Z}$, $s \in S$ and primes p.*

Proof. Since S is finitely generated, there is a positive integer k such that $ks \in \langle X \rangle$ for all $s \in S$. Let p be a prime dividing k. Since the natural map $G \to G_p$ sending every $g \in G$ to $g_p = g \otimes 1$ preserves p-heights, the set $X_p = \{x_p : x \in X\}$ is a decomposition basis for G_p and we have $S_p \cap \langle X_p \rangle = \langle S_p \cap X_p \rangle$. Assuming $y_p \in S_p$, there is a positive integer m such that $my \in S$. But then $my \in S \cap \langle X \rangle = \langle S \cap X \rangle$ and therefore $y \in S \cap X$, contradicting $y \notin S$. Thus $y_p \notin S_p$. By the local version of Lemma 6.8 for $\mathbb{Z}_p$-modules (see [**JL**, Lemma 4.4]) there is an $n_p \in \omega$ such that

$$|rp^{n_p} y_p + s_p|_p = \min\{|rp^{n_p} y_p|_p, |s_p|_p\}$$

for all $r \in \mathbb{Z}_p$ and $s \in S$. Now put $n = \prod_{p|k} p^{n_p}$ and let $s \in S$. It is clear that $|mny + s|_p = \min\{|mny|_p, |s|_p\}$ for all integers m whenever p divides k. Suppose p does not divide k. Then $ks \in S \cap \langle X \rangle = \langle S \cap X \rangle$ yields elements $x_1, \ldots, x_n \in S \cap X$ and $a_1, \ldots, a_n \in \mathbb{Z}$ such that $ks = a_1 x_1 + \ldots + a_n x_n$ and hence $s_p = \frac{a_1}{k}(x_1 \otimes 1) + \ldots + \frac{a_n}{k}(x_n \otimes 1)$. Since $\{x_1 \otimes 1, \ldots, x_n \otimes 1, y_p\}$ is a decomposition set, we obtain

$$|mny + s|_p = |mny_p + s_p|_p = \min\{|mny_p|_p, |s_p|_p\} = \min\{|mny|_p, |s|_p\}$$

as required. □

Warfield's local-global lemma will be needed.

Lemma 6.9 (Warfield [**War1**]). *Let A and B be abelian groups, S and T subgroups such that A/S and B/T are torsion, and $f : S \to T$ a homomorphism. Suppose for every prime p, the induced map $f_p : S_p \to T_p$ extends to a homomorphism $g(p) : A_p \to B_p$. Then f extends to a homomorphism $g : A \to B$ such that $g_p = g(p)$ for all primes p. If each map $g(p)$ is injective [bijective], then g is injective [bijective].*

We are now ready to prove the main result of this paper:

Theorem 6.10. *Let G and H be groups with partial decomposition bases and let δ be an ordinal. Suppose*

(1) *$\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ for all primes p and $\alpha < \omega\delta$;*
(2) *$\hat{w}_{\omega(\nu+1)}(c, p, e, G) = \hat{w}_{\omega(\nu+1)}(c, p, e, H)$ for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences and $\nu < \delta$;*
(3) *if $length(t(G_p)) < \omega\delta$, then $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.*

Then $G \equiv_\delta H$.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be partial decomposition bases for G and H as in Lemma 6.6. For $\nu \leq \delta$ let I_ν be the set of all maps $f : S \to T$ such that there are sets $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ with $f(X) = Y$ satisfying:

(i) S and T are finitely generated subgroups of G and H, respectively;
(ii) f is an $\omega\nu$-height-preserving isomorphism;
(iii) $X \subseteq S \subseteq \langle X \rangle^0$ and $Y \subseteq T \subseteq \langle Y \rangle^0$;
(iv) $U(x)$ and $U(f(x))$ are $\omega\nu$-compatible for every $x \in X$.

To prove that $G \equiv_\delta H$, we will show that the system $\{I_\nu : \nu \leq \delta\}$ satisfies condition (2) of Karp's Theorem 3.1. Suppose $f \in I_{\nu+1}$ where $\nu < \delta$, say $f : S \to T$ with associated $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and let $x \in G \backslash S$. To find an extension $g \in I_\nu$ of f with $x \in domain(g)$, we will show

(A) If x has a multiple in S, then there is such a map $g \in I_\nu$

and

(B) If $X \cup \{x\} \in \mathcal{C}$, then there is a map $g' \in I_{\nu+1}$ extending f such that $rx \in domain(g')$ for some positive integer r.

Then repeated application of (B) followed by an application of (A) yields an extension $g \in I_\nu$ of f with $x \in domain(g)$. To prove (A), suppose $rx \in S$ for some positive integer r. In order to construct the map g, we will apply Warfield's Lemma 6.9 to the groups $A = \langle S, x \rangle$ and $B = T^0$, so let p be a prime and consider the induced map $f_p : S_p \to T_p$. Since the natural map $G \to G_p$ preserves p-heights, the modules G_p and H_p have induced partial decomposition bases. Let $\alpha < \omega\delta$, $\nu < \delta$ and e an equivalence class of Ulm sequences. Then $\hat{u}(\alpha, G_p) = \hat{u}(\alpha, H_p)$ by [**F2**, Part 2, Lemma 16].

Let C be a set of representatives of the $\omega(\nu+1)$-compatibility classes, one for each class. By Corollary 4.3, $\hat{w}(e, G_p) = \min\{\sum_c \hat{w}(c, p, e, G), \omega\}$. So

$$\begin{aligned}
\hat{w}_{\omega(\nu+1)}(e, G_p) &= \min\{\sum_{e' \sim_{\omega(\nu+1)} e} \hat{w}(e', G_p), \omega\} \\
&= \min\{\sum_{e' \sim_{\omega(\nu+1)} e} \min\{\sum_c \hat{w}(c, p, e', G), \omega\}, \omega\} \\
&= \min\{\sum_{e' \sim_{\omega(\nu+1)} e} \sum_{c \in C} \sum_{c' \sim_{\omega(\nu+1)} c} \hat{w}(c', p, e', G), \omega\} \\
&= \min\{\sum_{c \in C} (\sum_{e' \sim_{\omega(\nu+1)} e} \sum_{c' \sim_{\omega(\nu+1)} c} \hat{w}(c', p, e', G)), \omega\} \\
&= \min\{\sum_{c \in C} \hat{w}_{\omega(\nu+1)}(c, p, e, G), \omega\} \\
&= \min\{\sum_{c \in C} \hat{w}_{\omega(\nu+1)}(c, p, e, H), \omega\} = \hat{w}_{\omega(\nu+1)}(e, H_p).
\end{aligned}$$

Now let $\mathcal{C}_{G_p}$ and $\mathcal{C}_{H_p}$ be the induced partial decomposition bases of G_p and H_p as in Lemma 5.2 and notice that the map f_p with associated sets $\{x \otimes 1 : x \in X\} \in \mathcal{C}_{G_p}$ and $\{y \otimes 1 : y \in Y\} \in \mathcal{C}_{H_p}$ satisfies the conditions (i)-(iii) as stated in the outlined proof of Theorem 5.3. Then by Theorem 5.3, f_p satisfies condition (2)(b) of Karp's Theorem 3.1, hence it can be extended to an $\omega\nu$-height-preserving isomorphism $g(p)$ with $x_p \in domain(g(p))$. By Lemma 6.9 we have a homomorphism $g : A \to B$ where $g(x) = y$ for some $y \in B$ and $g_p = g(p)$ for all p. Each map $g(p) : A_p \to B_p$ is injective and $\omega\nu$-height-preserving, therefore $g : \langle S, x \rangle \to \langle T, y \rangle$ is an $\omega\nu$-height-preserving isomorphism. Then g with associated sets X and Y satisfies conditions (i)-(iv), hence $g \in I_\nu$.

To verify (B), assume that $X \cup \{x\} \in \mathcal{C}$. By condition (iv) we have

$$\hat{w}_{\omega(\nu+1)}(c, p, e, \langle X \rangle^0) = \hat{w}_{\omega(\nu+1)}(c, p, e, \langle Y \rangle^0)$$

for all compatibility classes c of Ulm matrices, primes p and equivalence classes e of Ulm matrices. By Lemma 6.7, there is an element $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$ and

$$\hat{w}_{\omega(\nu+1)}(c, p, e, \langle X \cup \{x\} \rangle^0) = \hat{w}_{\omega(\nu+1)}(c, p, e, \langle Y \cup \{y\} \rangle^0)$$

for all c, p and e where $U(x) \sim_{\omega(\nu+1)} U(y)$ and $U_p(x) \sim_{\omega(\nu+1)} U_p(y)$ for all primes p. Then there are positive integers k and l such that $U_p(kx) =_{\omega(\nu+1)} U_p(ly)$ for all primes p. Let $x' = kx$ and $y' = ly$. Now proceed as in the proof of the classification in $L_{\infty\omega}$ (see [**J2**, Theorem 14]): Letting $\tilde{X} = X \cup \{x'\}$, we have $S \cap \langle \tilde{X} \rangle = \langle S \cap \tilde{X} \rangle$, so we can apply Lemma 6.8 to the group $\langle S, x'\rangle^0$ with decomposition basis $\tilde{X}$ and the subgroup S, and similarly to $\langle T, y'\rangle^0$, $Y \cup \{y'\}$ and T. Then there is a positive integer n such that

$$|mnx' + s|_p = \min\{|mnx'|_p, |s|_p\} \text{ and } |mny' + t|_p = \min\{|mny'|_p, |t|_p\}$$

for all $m \in \mathbb{Z}$, $s \in S$, $t \in T$ and primes p. Finally, let $S' = \langle S, nx'\rangle$ and $T' = \langle T, ny'\rangle$. Then f extends to the map

$$g' : S' \to T'$$

by sending nx' onto ny'. It is clear that g' is $\omega(\nu + 1)$-height-preserving. Let $X' = X \cup \{nx'\}$ and $Y' = Y \cup \{ny'\}$. Then $X' \subseteq S' \subseteq \langle X'\rangle^0$ and $Y' \subseteq T' \subseteq \langle Y'\rangle^0$, therefore g' is a map in $I_{\nu+1}$ with associated sets $X' \in \mathcal{C}$ and $Y' \in \mathcal{D}$ such that $nx' \in domain(g')$.

Consequently, f extends to a map $g \in I_\nu$ with $x \in domain(g)$, as desired. By symmetry, the conditions of Theorem 3.1(2) are satisfied and it follows that $G \equiv_\delta H$. □

Acknowledgment. The authors wish to thank the referee for helpful comments and suggestions.

References

[BE] J. Barwise and P. C. Eklof, *Infinitary properties of abelian torsion groups*, Ann. Math. Logic **2** (1970), 25-68. MR0279180 (43:4906)

[F1] L. Fuchs, *Infinite Abelian Groups*, Vols. **I** and **II**, Academic Press, New York (1970 and 1973). MR0255673 (41:333)

[F2] L. Fuchs, *Abelian p-Groups and Mixed Groups*, University of Montreal Press (1980). MR569744 (82f:20081)

[GLLS] R. Göbel, K. Leistner, P. Loth and L. Strüngmann, *Infinitary equivalence of $\mathbb{Z}_p$-modules with nice decomposition bases*, J. Comm. Algebra **3** (2011), 321-348. MR2826477

[H] P. Hill, *On the classification of abelian groups*, 1967 (unpublished).

[HR] R. Hunter and F. Richman, *Global Warfield groups*, Trans. Amer. Math. Soc. **266** (1981), 555-572. MR617551 (82k:20091)

[J1] C. Jacoby, *Abelian groups with partial decomposition bases in $L_{\infty\omega}$*, to be submitted for publication.

[J2] C. Jacoby, *The classification in $L_{\infty\omega}$ of groups with partial decomposition bases*, PhD thesis 1980, University of California, Irvine, revised version (2010).

[JL] C. Jacoby and P. Loth, *$\mathbb{Z}_p$-modules with partial decomposition bases in $L^\delta_{\infty\omega}$*, to be submitted for publication.

[K] C. Karp, *Finite quantifier equivalence*, The Theory of Models, North-Holland, Amsterdam (1965), pp. 407-412. MR0209132 (35:36)

[L] P. Loth, *Classifications of Abelian Groups and Pontrjagin Duality*, Algebra, Logic and Application Series, Volume 10, Gordon and Breach Science Publishers, Amsterdam (1998). MR1741074 (2001c:20117)

[R] P. Rothmaler, *Introduction to Model Theory.* Prepared by Frank Reitmaier, translated and revised from the 1995 German original by the author. Algebra, Logic and Application Series, Volume 15, Gordon and Breach Science Publishers, Amsterdam (2000). MR1800596 (2001h:03002)

[St] R. O. Stanton, *Almost-affable abelian groups*, J. Pure Appl. Algebra **15** (1979), 41-52. MR532962 (80i:20029)

[Sz] W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. **41** (1954), 203-271. MR0072131 (17:233e)

[U] H. Ulm, *Zur Theorie der abzählbar-unendlichen abelschen Gruppen*, Math. Ann. **107** (1933), 774-803. MR1512826

[Wal] E. A. Walker, *Ulm's theorem for totally projective groups*, Proc. Amer. Math. Soc. **37** (1973), 387-392. MR0311805 (47:367)

[War1] R. Warfield, *Classification theory of abelian groups I: Balanced projectives*, Trans. Amer. Math. Soc. **222** (1976), 33-63. MR0422455 (54:10444)

[War2] R. Warfield, *Classification theory of abelian groups II: Local theory*, Lecture Notes in Math., Vol. 874, Springer Verlag, New York (1981), pp. 322-349. MR645938 (83h:20052)

Jacoby Consulting, Long Beach, California
E-mail address: cjacoby@jacobyconsulting.com

Fakultät für Mathematik, Universität Duisburg-Essen, D 45117 Essen, Germany
E-mail address: katrin.leistner@uni-duisburg-essen.de

Department of Mathematics, Sacred Heart University, Fairfield, Connecticut 06825, USA
E-mail address: lothp@sacredheart.edu

Fakultät für Mathematik, Universität Duisburg-Essen, D 45117 Essen, Germany
E-mail address: lutz.struengmann@uni-duisburg-essen.de