2008

Is Mathematics Created by Humans or is it Discovered by Humans? A Catholic Intellectual Perspective

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Introduction

I have taught at a Catholic University for eight years. I have been a practicing Catholic my entire life. Over the past few years, my university has become more conscious of the Catholic Intellectual Tradition (CIT). I was relieved by this because I saw this as an opportunity for my intellectual life and my spiritual life to interact and improve upon one another. Over the past two years, I have done significant studying of the CIT. I took a year-long seminar offered by the university, participated in Collegium in the summer of 2007, and have done some reading on the subject. While I am intrigued by what I have learned about the CIT, I am at the same time disappointed. The reason for my disappointment is simple. In my studying of the CIT, the CIT is often discussed in light of the sciences such as biology, chemistry, and physics, or in light of the arts such as literature, art, and music. However, the CIT is rarely discussed in light of my discipline – mathematics. In fact, I have seen the discussion of the CIT to go so far as to reduce the discipline of mathematics to merely a set of skills that is used in the sciences. In this essay, I intend to show that not only is it appropriate to discuss the CIT in light of mathematics, the CIT can actually be exemplified in mathematics!

One of the main pillars of the CIT is that knowledge is sacramental. Learning about the world is a way of encountering God. Everything and everyone in the world can reveal God.[1] Thus it is natural to discuss the CIT in light of the sciences because by making scientific discoveries, we are in turn discovering God. As said in Genesis, "In the beginning God created all things…" and later in Genesis, "God saw that it was good." Thus it is the nature of humans to want to reveal God, to learn about the unknown, and to be inquisitive. This desire has been implanted in our souls. The mind has the desire for knowledge like the stomach has the desire for food.[2] As a Catholic intellectual, it is my job to discover and to realize that my discoveries are revealing the face of God.

It is also natural to discuss the CIT in light of the arts such as art, music, or literature. Unlike the sciences, these are not created by God but rather are the products of human thought. However, we should keep in mind that these disciplines reflect everyday world situations and events. But since the everyday world is governed by a higher spirit, this immediately makes the arts sacramental. Keeping with the idea that these disciplines are the product of human thought, they may not necessarily reflect reality, but they may inspire what should be reality. The arts are a manifestation of beauty – beauty which is ultimately authored by God.[3] So it is therefore natural to study the arts in light of the CIT.

Having been involved in mathematics my entire life, I can understand why many would not think of mathematics as a discipline that involves discovery like the sciences, or as a subject of beauty like the arts. After all, mathematics is usually taught in elementary school through high school, and even college in some instances, as a series of

rules and formulas for manipulating numbers and equations. These rules and formulas are usually presented as facts with no explanation as to why they are true or how they are derived or discovered. Thus much of the population has never been exposed to the discovery nature of mathematics. Further, it is in the discovery of mathematics that we see its intrinsic beauty. Thus when a student is deprived of learning how mathematics can be discovered, the student is simultaneously being deprived of the beauty of mathematics as one of the most fascinating products of human thought.

In this essay, I intend to illustrate how mathematics can be studied in light of, and even enhanced by, the CIT. I do this in two ways. In the next section, I discuss the main ideas of how mathematics is created and discovered. The ideas are illustrated with two well-known mathematical theorems. To put the reader at ease, I do not assume the reader has any mathematical background. In the following section, I offer some reflections on the teaching of mathematics and why I feel that teaching it from a Catholic intellectual perspective enhances the subject. I use the teaching of geometry as an example, but again stress that I do not assume any mathematical background on the part of the reader.

Creating and Discovering Mathematics in Light of the CIT

To understand the nature of mathematics, it is beneficial to understand the basic history of mathematics. Numbers were created by humans in order to satisfy their needs. The need to count is an everyday need thus came about the natural numbers 1, 2, 3, etc. To account for nothingness, we have the number zero. However, there are instances where one loses more than is gained, hence the need for negative numbers. So at this point, we now have the integers ...., -3, -2, -1, 0, 1, 2, 3,.... Yet there are times in everyday life where we are concerned with only a portion of something, hence came the factions such as ½ or ⅔. We note at this point that a fraction is always written as an integer over another integer. The integers together with the factions are termed the rational numbers. (Actually an integer can be written as fraction by making the denominator one, e.g. $7={}^{7}/_{1}$.) It is tempting at this point to believe that humans have created all of the numbers that are needed. However, we are indeed missing some. For example, suppose you wanted to find the positive number x such that $x^2=2$. This number would be $\sqrt{2}$ which is unable to be expressed as a fraction, thus giving rise to the existence of irrational numbers. The rational numbers and irrational numbers together make up the real numbers. The term real numbers is actually oxymoronic in that there is nothing "real" about them. Numbers are solely the product of human thought that were created by humans to satisfy everyday needs. (We should note that there are also imaginary numbers too, but that is beyond the scope of this essay.) At this point we are beginning to realize that mathematics can be seen as the product of human thought, but since the need for mathematics arises out circumstances not necessarily created by humans, it is certainly plausible that there is a discovery element to mathematics as well.

Not only were numbers created for use in everyday life, operations between numbers were also created. The need in everyday life to add, subtract, multiply, and divide is clear. Focusing on the operation of division, it is often hopeful in everyday life that we can divide a quantity evenly. Therefore, we define a natural number a to be a *divisor* of the natural number b if a/b does not leave a remainder. For example 4 is a divisor of 12 since 12/4 does not leave a remainder; yet 4 is not a divisor of 10 since 10/4

leaves a remainder. Since we merely created a definition, we are still at the point that mathematics is the product of human thought. At this point, it is natural to create another definition, that of a prime number. A *prime number* is a natural number (excluding one) that has no divisors other than one and itself. For example, 2 is a prime number because its only divisors are 1 and 2. Likewise 3 is a prime number because its only divisors are 1 and 3. However, 4 is not a prime number because not only are 1 and 4 divisors, so is 2 as 4/2 does not leave a remainder. We can also see that 5, 7, 11, and 13 are prime numbers. However 6 is not a prime number (its divisors are 1, 2, 3, and 6), and neither is 8 (its divisors are 1, 2, 4, and 8). A natural number that is not prime is said to be *composite*.

So far we have only considered the definition-creating aspect of mathematics. While mathematical definitions are solely the product of human thought, these definitions often lead to fascinating questions. For example, the definition of a prime number begs the question: Are there infinitely many prime numbers? Now that we are able to ask questions about mathematics, we now see that there is truly a discovery element to mathematics. In order to answer this question, we must use reason. Keeping in mind that faith and reason are pillars of the CIT,[4] we can now begin to surmise that mathematics can be viewed from a Catholic intellectual perspective. From a Trinitarian perspective, discovery is to keep in mind what you know, but more importantly to know what you don't know.[5] So before definitively answering the question at hand, let's observe what we know. To this end we consider two schools of thought. First, there is the school of thought that since there are infinitely many numbers, there is a likely chance that there are infinitely many primes. However, another school of thought is that as numbers get larger, there are more possible numbers less than it that could be divisors of it. Hence there could come a point where the numbers are so large that there are enough numbers below it to guarantee that a divisor exists. The second school of thought has merit. After all, there are twenty-five prime numbers below 100, but only twenty-one prime numbers between 100 and 200, and only sixteen prime numbers between 200 and 300. The prime numbers seem to appear with less frequency as the natural numbers increase thus making it plausible that at some point they cease all together.

When researching mathematics, using evidence of what we know is a desirable start, especially in light of the CIT. One aspect of the CIT that is exemplified in the studying of mathematics is that we let things reveal themselves and speak to us on their own terms. We pay attention to see and know what is true, and we are impressed by its beauty.[6] So far, we have seen an interesting, albeit beautiful, structure to the natural numbers. The prime numbers seem to appear randomly but thin out as the natural numbers increase. Thus in the process of answering our question about the infinitude of prime numbers, we are conducting a search for truth which is a process of discovery.[7] So in a continued effort to answer the question at hand, let's make further discoveries about our number system. For example, if a number is composite then by definition it can be factored as the product of two natural numbers other than one and itself (e.g. $300=3*100$). However, if either of these numbers is composite, it in turn can be factored as the product of two natural numbers (e.g. $300=3*100=3*2*50$). We can continue this process until we have our original number written as the product of only prime numbers (e.g. $300=2*2*3*5*5$). So another beauty of our number system has been revealed further as we can now more accurately define a prime number as a number that has no *prime* divisors.

Another discovery we make occurs by considering a new number created by adding 1 to a composite number. Observe that the prime divisors of that composite number will not be prime divisors of our new number because the remainder will be 1 when you divide the new number by such a prime divisor. For example, 2 is a prime divisor of 4; but 2 is not a prime divisor of 5 as 5/2 leaves a remainder of 1. Likewise, 2 is a prime divisor of 6, but 2 is not prime divisor of 7 because 7/2 leaves a remainder of 1. Also, 3 is a prime divisor of 6, but 3 is not prime divisor of 7 because 7/3 leaves a remainder of 1. At this point, it *appears* that adding one to a composite number will yield a prime number. So have we discovered a way to produce infinitely many primes thus answering our question concerning the infinitude of primes? Unfortunately we have not. Consider the composite number 8. We see that 2 is a prime divisor of 8, hence 2 is not a prime divisor of 9 as 9/2 leaves a remainder of 1. But 9 is not prime because it has 3 as a prime divisor. So even though this approach to finding the location of prime numbers was unsuccessful, we are further seeing that there is an inherent patter and beauty to our number system. As a Catholic intellectual, this never surprised me. Catholic intellectuals are convinced that the world is "not random or disordered. It came into being not by chance or spontaneity, but by God's wisdom and love."[8]

Despite the fact that our most recent approach to answer the question at hand was unsuccessful, the Catholic intellectual realizes that the early successes we obtained from our approach have the potential to lead us to an alternate discovery. Let's modify our approach by adding 1 to composite numbers whose prime divisors are distinct and consecutive beginning with 2. At first glance, this looks more promising:

2*3=6. Observe that 6 + 1 = 7 which is prime.
2*3*5=30. Observe that 30 + 1 = 31 which is prime.
2*3*5*7=210. Observe that 210+1 = 211 which is prime.
2*3*5*7*11=2310. Observe 2310+1=2311 which is prime.

So now have we found a method of producing infinitely many primes? Unfortunately, we have not. If we proceed just one more step, we see that 2*3*5*7*11*13=30030, yet 30030 + 1 = 30031 is not prime since 59 and 509 are prime divisors of 30031. So again we have made another failed attempt to answer the question at hand. However, in the process of failing, another discovery has been made! Notice that the prime divisors of 30031 are far above largest prime divisor of 30030 (which is 13). So perhaps while the number we create by adding one to the product of consecutive prime numbers may not be prime, it may lead us to the whereabouts of other prime numbers. Using this idea, it is beginning to become more likely that there are infinitely many prime numbers. To prove this outright, we will assume there are finitely many primes and deduce a contradiction. If there were only finitely many primes, then there would be a largest prime number, call it P. Now consider the number N created by adding one to the product of all of the prime numbers i.e. $N=(2*3*5*7*11*....*P)+1$. Clearly N is larger than the largest prime P, thus by our assumption N cannot be prime. However, N does not have any prime divisors because, by an earlier observation, the remainder will always be 1 when divided by any prime number 2, 3, 5, 7, 11, ..., P. So by definition, since N has no prime divisors, it must be prime. But we had just said that N cannot be prime. Hence we have deduced a

contradiction. Thus our original assumption that there are finitely many primes must be false. Thus we can definitively say that there are infinitely many prime numbers.

This is a prime example (no pun intended) of how mathematics is discovered. Note that the discovery began with a definition of prime numbers – a definition that was a creation from human thought. Once the definition was created, beautiful patterns within our number system began to reveal themselves. So this begs the question: is mathematics created by humans or is it discovered by humans? We have already seen that mathematics is the search for truth. However, in mathematics, in order to understand what truth we are searching for, humans need to create pertinent definitions and axioms. Once this occurs, we can ask questions, or better yet, the questions arise naturally. According to the CIT, asking questions and desiring knowledge is the premise of what it means to be human.[9] This has been true as far back as Adam and Eve and their struggle with the tree of knowledge. The temptation of being human is to know what God knows. Much of what is means to be human is to formulate questions based on what we already know. This is not only a major idea of the CIT, but it is the backbone of mathematics. For example, we already know that there are infinitely many prime numbers. Yet when looking at the prime numbers more closely, we see many pairs of *twin primes,* i.e. two prime numbers whose difference is two such as 11 and 13, 17 and 19, and 2549 and 2551. Thus we are now tempted to ask another question to further our knowledge of prime numbers, namely: Are there infinitely many pairs of twin primes? To this date, that question remains unanswered![10]

Most people often regard mathematics as a science because it is used in the sciences to describe the natural world. However, mathematics is not a science in itself. We have seen from the example of prime numbers that mathematics is often divorced from the natural world. This is because mathematics is a discovery of something that is the product of human thought rather than a discovery of the natural world. Mathematics is also used to describe the arts. Since art is a way of encountering God, this topic is worth investigating. The arts are created by humans as an expression of the way things should be.[11] While opinions may differ, there is often a consensus of what is aesthetically pleasing. For example, if we draw a map consisting of several regions, it is aesthetically pleasing to have regions that share a common border (not just a common point) to be of a different color. For example, in a map of the continental United States, we would want California and Nevada to be of different colors because they border each other. As with the example involving prime numbers, we are creating a definition, namely what it means to be aesthetically pleasing. Now that we have a definition, the definition has the potential to lead us to many questions. For example, how many colors are required in order to color a map of the continental United States so that it is aesthetically pleasing? Clearly, since there are 48 states in the continental United States, we can use 48 different colors, i.e. one for each state, thus the map will be colored in an aesthetically pleasing fashion. However, a more interesting question is: What is the *fewest* number of colors required to color a map of the continental United States in an aesthetically pleasing way? Observing that Connecticut, Massachusetts, and Rhode Island each border the other two states, we see that we need at least three colors. We don't have to continue much further before a fourth color is necessitated. But after introducing a fourth color, we see that it is possible to color the rest of the map without

the need for a fifth color. Thus only four colors are required to color a map of the continental United States in an aesthetically pleasing way.

At this point it becomes natural to ask the question: what is the fewest number of colors required in general to color a map in an aesthetically pleasing way? In order to discuss how this question may be answered from a Catholic intellectual perspective, we need to note some items. First, it should be noted that this question was first asked in 1852 by Francis Guthrie long before there was a continental United States.[12] We used the continental United States in the previous paragraph because it is a convenient frame of reference for most readers. Second, in our process of letting the mathematics reveal itself as is desirable in the CIT, we first surmise that this question may likely depend on what map we are considering. To some extent this is true. There are maps that require less than four colors. The map in Figure 1 only requires two colors; the map in Figure 2 only requires three colors.

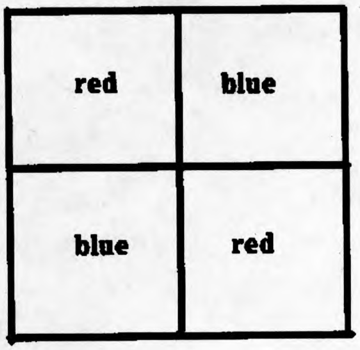


Figure 1

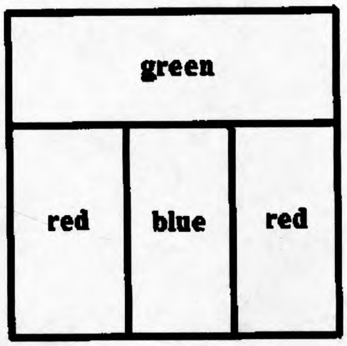


Figure 2

Guthrie observed that there are many maps that require four colors. However, he was unable to find a map that required five or more colors. Thus it was conjectured, but not proven, that no map requires more than four colors to color it in an aesthetically pleasing way. This became known as the Four Color Conjecture.[13]

Map coloring is part of a field of mathematics known as graph theory. When researching this question, mathematicians converted maps into what are known as *planar graphs* because previous mathematical research had already been conducted on planar graphs. Over the next several decades, mathematicians further investigated the mathematical structure of planar graphs and many discoveries were revealed. In 1890, it was proven that no map requires more than five colors in order to be colored in an aesthetically pleasing way. This was a great first step because it eliminated the possibility of a map requiring six or more colors. However, since no map requiring five colors had been discovered, the goal now was to either find such a map or prove that no such map exists. Further research continued on this question. In 1970, it was proven that no map with 25 or fewer regions would require a fifth color. A couple years later, it was shown that no map with 95 or fewer regions would require a fifth color. Finally, mathematicians Kenneth Appel and Wolfgang Haken determined that any map must

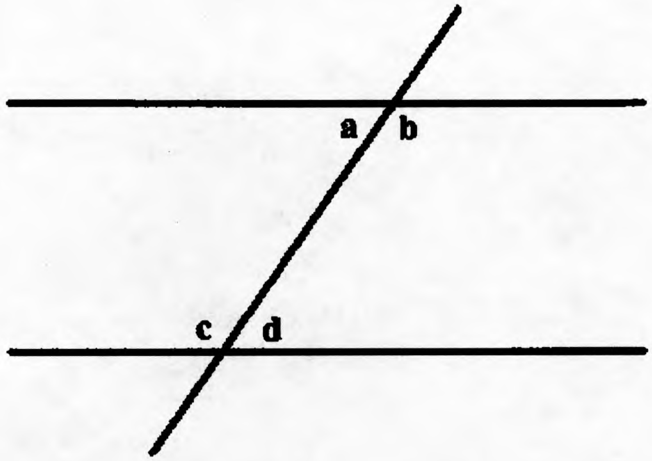
contain at least one of 1,936 configurations. In 1976, Appel and Haken used computing power to test each of these 1,936 configurations and determined that each of these configurations was four-colorable. Thus after 124 years, it was finally proven that no map requires more than four colors in order to be colored in an aesthetically pleasing way.[14]

In the CIT, learning is valued for its own sake.[15] The fact that there are infinitely many prime numbers and the fact that all maps require no more than four colors don't necessarily have any deep practical applications in "the real world." Rather the value of having this knowledge is simply to have it. Mathematics contributes to the human "treasury of knowledge"[16] just as equally as any discipline in the sciences or any discipline in the arts. What makes the discipline of mathematics unique is that it has attributes in common with both the sciences and the arts. Mathematics has the creation aspect to it similar to that of the arts, but also has a discovery aspect to it similar to that of the sciences.

A CIT Perspective on Teaching Mathematics

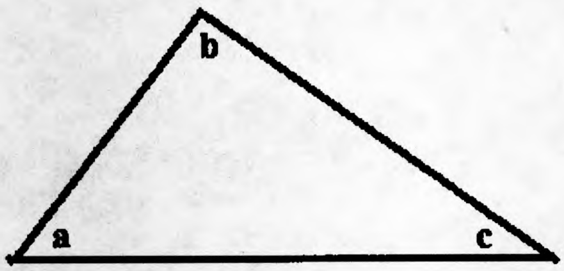
The way that one views mathematics will have a direct impact on he or she teaches and researches it. As I reflect over the past eight years of my career, I see that being a Catholic intellectual has had a profound influence on my teaching and researching of mathematics. In a Catholic university, or any university for that matter, teaching and research are seen as two ends of a spectrum. There are institutions that place a greater value on research and place their resources into hiring academics who are more research oriented in their careers. On the other end of the spectrum, many institutions place a greater emphasis on teaching and thus place their resources accordingly. As a Catholic intellectual, I don't necessary view teaching and research as being disjoint. To teach mathematics in light of the CIT, one needs to teach the students how to be researchers. In other words, it is not sufficient to merely dictate a bunch of facts and theorems to a class. The role of a Catholic mathematics teacher is to aid students into gaining insight as to how these facts and theorems were discovered. As Saint Augustine stated, "A teacher who repeatedly says 'believe me' without explaining why things must be so soon forfeits authority and blocks the path to understanding."[17] Saint Aquinas complemented this in his statement, "If the teacher determines the question by appeal to authorities only, the student will be convinced but will not acquire understanding and will go away with an empty mind."[18] In the previous section of this essay, we resolved that definitions are created but the mathematics is discovered. This ideology needs to permeate a Catholic mathematics teacher.

I find that the most effective way to teach mathematics in light of the CIT is to show how discovering one theorem can lead to the discovery of more advanced theorems. Teaching can be viewed a collaborative research with students where you are guiding the students through the discovery process. Let's illustrate this with a basic example in teaching geometry. When teaching geometry, we can easily point out to the students that given two parallel lines with a nonparallel line crossing through them, the alternate interior angles will each have the same measure. Thus in the drawing below, angles a and d have the same measure as do angles b and c.

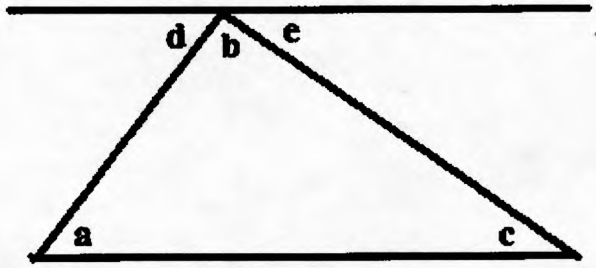


Before continuing, we need to define a way to quantify precisely what we mean by an angle measurement. We will use degrees as the unit of measure and say that a right angle has 90 degrees. Observe that the way in which we defined the measurement of an angle is a human creation, not a discovery. With this creation observe that measures of adjacent angles forming a straight line sum to 180 degrees, i.e. the measures of angles a and b sum to 180 degrees as do the sum of the measures of angles c and d.

In geometry, students are often taught that the sum of the angles of a triangle is 180 degrees, but they are not taught why this is true. Hence the student is likely to believe that this fact was merely invented. This illustrates an issue I have with math education because students then fail to distinguish between what is created by humans and what is discovered by humans. To assist students in discovering why the angles of a triangle sum to 180 degrees, first observe the following triangle:

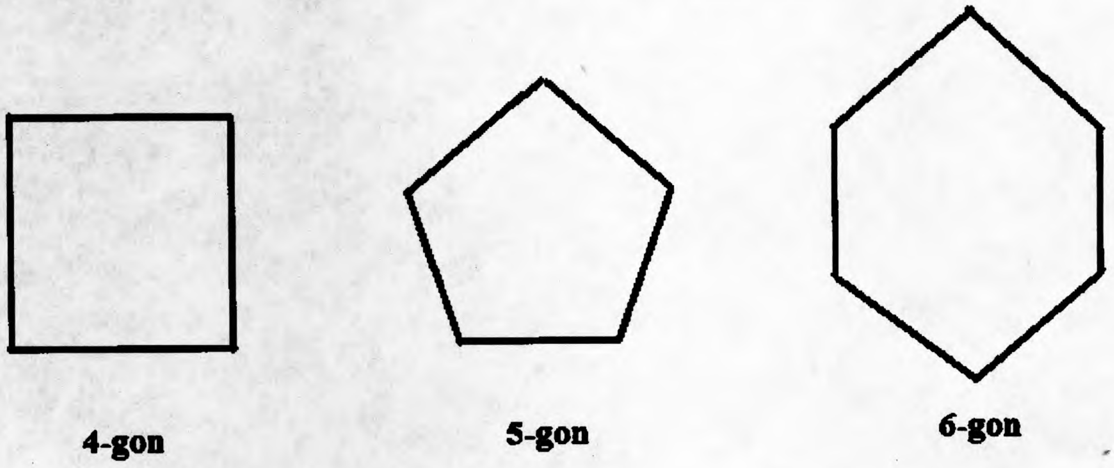


Our goal is to prove that angles $a+b+c=180$. To this end, we draw a line parallel to line ac of the triangle that just touches the point of angle b. Label the angles adjacent to angle b as angles d and e.

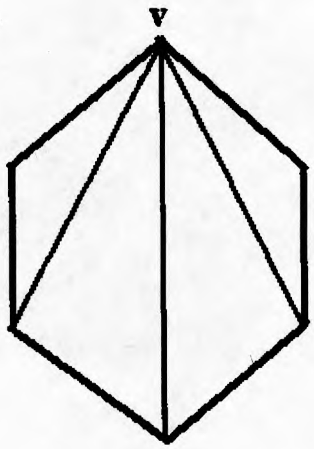


Since *d, b,* and *e* are adjacent angles which together form a straight line, we remind the students that it follows that $d+b+e=180$. However, since angles *a* and *d* are alternate interior angles with respect to line *ab* of the triangle, we also remind the students that angles *a* and *d* have equal measure. Similarly, since angles *c* and *e* are alternate interior angles with respect to the line *bc* of the triangle, it follows that angles *c* and *e* also have equal measure. Thus in the equation $d+b+e=180$, we can replace *d* with *a* and *e* with *c* to obtain $a+b+c=180$ which is precisely what we wanted to show.

At this point, we took two observations - alternate interior angles have the same measure and the measures of adjacent angles forming a straight line sum to 180 degrees – and used these observations to discover the theorem that the sum of the angles of a triangle is 180 degrees. Since the students were a part of the process of discovery, this theorem will have more meaning to them. This is the ultimate outcome of using the CIT in one's teaching. To enlighten students further of the CIT by impressing upon them that one theorem is often used to make further discoveries, we will use this theorem to prove a more beautiful theorem. We define an *n-gon* as an enclosed shape formed by *n* straight line segments. Thus below we have an example of a 4-gon, 5-gon, and 6-gon:



Another theorem that is often learned in geometry is that the sum of the degrees of the angles of an *n*-gon is $180*(n-2)$. Again, have the students learned and understood the theorem, or have they just memorized it? To aid in student understanding and learning, the students can be told to observe that for any *n*-gon, we can pick a vertex *v* and then draw *n-3* lines joining that vertex to each other vertex of the *n*-gon that *v* is not already connected to. See the example below with the 6-gon:



Observe that this divides the *n*-gon into *n-2* triangles. However, we already discovered that the sum of the angles of a triangle is 180 degrees. So since the sum of the angles of an *n*-gon is precisely the sum of the angles of the *n-2* triangles that were created, it follows that the sum of the angles of an *n*-gon are *180*(n-2)*.

When teaching mathematics where you assist the students in discovery, the students are able to understand the treasures of mathematics as opposed to merely memorizing seemingly random facts. In light of the CIT, the students are getting a glimpse of God and seeing that the world God created is not random nor disordered,[19] but rather that God is anxious for humans to reveal the world He created and consequently, reveal Him. This is precisely what research is. Therefore, a Catholic mathematics teacher should be teaching the students how to research mathematics. It is through this that students truly learn the true beauty of mathematics and the world of hidden treasures that mathematics has to offer.

Concluding Thoughts

Numbers and other mathematical structures are exclusively the product of human thought. Humans created mathematics for several reasons: to satisfy their everyday need to quantify things, to gain a greater appreciation for the arts, but also to be studied for the mere sake of learning. Mathematics is about exploring and discovering ideas that are created by humans. Once humans create a mathematical object and supply the necessary definitions and axioms, the object takes on a life of its own. The object often leads to questions for pondering, theorems to be discovered, and many surprising hidden treasures to be found. A simple definition of a prime number leads us to discover that there are infinitely many prime numbers. The definition of what is aesthetically pleasing regarding map coloring leads to the surprising result that one never needs more than four colors to color a map, no matter how large the map! From a Catholic intellectual perspective, mathematics is truly sacramental because by discovering mathematics, we are discovering the products of human thought. Since as Catholics we learn from Genesis that humans are made in the image of God, it follows that mathematics is the ultimate search for truth.

The CIT stresses passionate learning and doing work for the common good. Unfortunately, it has never been well known that mathematics stresses these same ideals. Many of the theorems that are proven are proven using earlier proven theorem. Thus the population of mathematicians is a society in which we use each others' work to gain further knowledge about our own field, hence a pursuit for the common good. Work for the common good can also be seen in mathematics through the vastly disparate branches within the discipline. In this paper, we have dealt with three such branches: number theory (the prime number problem), graph theory (the map coloring problem), and geometry. There are numerous other branches of mathematics as well such as topology, analysis, modern algebra, differential equations, just to name a few. The mathematics student who aspires to be a mathematician chooses an area of specialization and dedicates his or her career to contributing to that area of mathematics. In this respect, the mathematics field can be seen as a microcosm of the human community that God has created. In society, we need people of all different types of specialization in order to contribute to the common good. We need teacher, lawyers, doctors, plumbers, mechanics, etc. Similarly in mathematics we need geometers, algebraists, number theorists, analysts, etc. in order to contribute to the common good within mathematics and to reveal the many faces of God which is the responsibility of all intellectuals.

Notes

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