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Abelian groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$, Part II

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Dedicated to Professor Rüdiger Göbel in honor of his 70th birthday

Abstract. We consider abelian groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$ for ordinals δ. Jacoby, Leistner, Loth and Strüngmann developed a numerical invariant deduced from the classical global Warfield invariant and proved that if two groups have identical modified Warfield invariants and Ulm-Kaplansky invariants up to $\omega\delta$ for some ordinal δ, then they are equivalent in $L^{\delta}_{\infty\omega}$. Here we prove that the modified Warfield invariant is expressible in $L^{\delta}_{\infty\omega}$ and hence the converse is true for appropriate δ.

1. Introduction

The classical theorem of Ulm [**U**] gives a complete classification of countable abelian p-groups in terms of numerical invariants, the *Ulm-Kaplansky invariants*. For uncountable p-groups this theorem is false; however, Hill [**H**] and Walker [**Wal**] were able to extend it to the class of totally projective p-groups, the largest natural class of abelian p-groups in which the Ulm-Kaplansky invariants distinguish between non-isomorphic groups. It was Szmielew [**Sz**] who first considered abelian groups from a model-theoretic point of view. Barwise and Eklof [**BE**] took up Szmielew's approach and investigated abelian p-groups of arbitrary cardinality in $L^{\delta}_{\infty\omega}$, leading to a generalization of Ulm's theorem and a characterization of $L_{\infty\omega}$-equivalence classes of torsion groups.

Warfield groups are defined to be direct summands of simply presented groups or, alternatively, are abelian groups possessing a nice decomposition basis with simply presented cokernel. By adding new invariants, the *Warfield invariants*, Ulm's theorem was generalized to Warfield groups by Warfield [**War**] in the p-local case and by Hunter, Richman [**HR**] and Stanton [**St**] in the global case. Generalizing Warfield groups, Jacoby [**J1**], [**J2**] defined abelian groups possessing a partial decomposition basis and was able to prove classification theorems in $L_{\infty\omega}$ in the global case and in $L^{\delta}_{\infty\omega}$ for modules over a complete discrete valuation ring. Independently, similar results were obtained by Göbel, Leistner, Loth and Strüngmann [**GLLS**] who considered $\mathbb{Z}_p$-modules with nice decomposition bases. In [**JLLS**], the authors together with Leistner and Strüngmann introduced invariants deduced from the global Warfield invariants and proved a classification theorem for abelian

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groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$ (see Theorem 4.1). In this paper, the converse of this result is proved for appropriate ordinals δ. More specifically, we show that the aforementioned invariants are expressible in $L^{\delta}_{\infty\omega}$ whenever $\delta = \omega\gamma$ and γ is a limit ordinal (Theorem 4.9). Consequently, the converse of Theorem 4.1 holds for those ordinals δ (Corollary 4.10).

For notation and terminology on abelian groups and on model theory, we may refer to the books [**F1**], [**F2**], [**L**] and [**R**].

2. Algebraic background

All groups considered in this paper are abelian. The reader should be familiar with the concepts of the p-height of an element x, written $|x|_p$, the length of a group A, written $l(A)$, the concept of a Warfield group and that of a decomposition set and decomposition basis. The height is used to define the Ulm-Kaplansky invariants of a group A, $u_p(\alpha, A)$ for each ordinal α and prime p. The global Warfield invariants $w(c, p, e, A)$ are defined for each compatibility class of Ulm matrices c, prime p and equivalence class of Ulm sequences e. These cardinal numbers, together with the Ulm-Kaplansky invariants, form a complete set of isomorphism invariants for Warfield groups (see Hunter-Richman [**HR**], Stanton [**St**]).

Barwise and Eklof's [**BE**] modified Ulm-Kaplansky invariant is defined as

$$\hat{u}_p(\alpha, A) = \min\{u_p(\alpha, A), \omega\}.$$

If we add the invariant $\hat{u}_p(\infty, A) = \min\{\dim(p^{\infty}A[p]), \omega\}$, these invariants classify all torsion groups in $L_{\infty\omega}$ (see [**BE**]). Because of this, a subgroup of particular interest is that of all torsion elements of A, which we denote $t(A)$. For a subgroup G of A, let $G^0 = \{a \in A : ra \in G \text{ for some nonzero integer } r\}$.

The class of groups considered in this paper consists of those with a partial decomposition basis [**J2**]. We say $\mathcal{C}$ is a *partial decomposition basis for* A if

(i) $\mathcal{C}$ is a nonempty collection of finite subsets of A,
(ii) if $X \in \mathcal{C}$, then X is a decomposition set, and
(iii) if $X \in \mathcal{C}$ and $x \in A$, then there is a $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $x \in \langle Y \rangle^0$.

Note that if $\mathcal{C}$ is a partial decomposition basis, so is $\{\{a_1x_1, \ldots, a_nx_n\} : a_1, \ldots, a_n \in \mathbb{Z}\backslash\{0\}$ and $x_1, \ldots, x_n \in X$ for some $X \in \mathcal{C}\}$, a fact we will use repeatedly.

We will need the following definitions "up to α" for an ordinal α.

For β and γ ordinals we define $\beta \geq_\alpha \gamma$ to mean $\beta \geq \gamma$ if $\gamma < \alpha$ and $\beta \geq \alpha$ if $\gamma \geq \alpha$. We write $\beta =_\alpha \gamma$ if

$$\min\{\beta, \alpha\} = \min\{\gamma, \alpha\}.$$

Then by $\beta >_\alpha \gamma$ we mean $\beta \geq_\alpha \gamma$ and $\beta \neq_\alpha \gamma$.

If $M = [m_{p,i}]$ and $N = [n_{p,i}]$ are Ulm matrices, we define $M \geq_\alpha N$ as $m_{p,i} \geq_\alpha n_{p,i}$ for all p and i, and other relations are defined similarly. We say $M \sim_\alpha N$ if and only if for some m and n, $mM \geq_\alpha N$ and $nN \geq_\alpha M$. If c is a compatibility class, we write $M \sim_\alpha c$ if $M \sim_\alpha N$ for some $N \in c$. We write $c \sim_\alpha c'$ to mean $M \sim_\alpha N$ for some $M \in c$ and $N \in c'$. Notice that $c \sim_\alpha c'$ if and only if there are $M = [m_{p,i}] \in c$ and $N = [n_{p,i}] \in c'$ such that $m_{p,i} =_\alpha n_{p,i}$ for all p and i (cf. [**JLLS**]).

In the same way, if (α_i) and (β_i) are Ulm sequences, we define $(\alpha_i) \geq_\alpha (\beta_i)$ if and only if for all i $\alpha_i \geq_\alpha \beta_i$. The other relationships are defined similarly. $(\alpha_i) \sim_\alpha (\beta_i)$ if and only if for some m and n, $\alpha_{i+m} =_\alpha \beta_{i+n}$ for all i. If e is an equivalence class of Ulm sequences, we write $(\alpha_i) \sim_\alpha e$ if $(\alpha_i) \sim_\alpha (\beta_i)$ for some $(\beta_i) \in e$ and $e \sim_\alpha e'$ if $(\alpha_i) \sim_\alpha (\beta_i)$ for some $(\alpha_i) \in e$ and $(\beta_i) \in e'$.

If A is a group with a partial decomposition basis $\mathcal{C}$, we define

$$\hat{w}(c, p, e, A) = \sup|\{x \in X : X \in \mathcal{C}, U(x) \in c \text{ and } U_p(x) \in e\}|$$

where c is a compatibility class of Ulm matrices, p a prime and e is an equivalence class of Ulm sequences. Notice that if A has a decomposition basis, then $\hat{w}(c, p, e, A) = \min\{w(c, p, e, A), \omega\}$. Jacoby, Leistner, Loth and Strüngmann [**JLLS**] generalized this definition up to α an ordinal:

$$\hat{w}_\alpha(c, p, e, A) = \min\{\sum_{e' \sim_\alpha e, c' \sim_\alpha c} \hat{w}(c', p, e', A), \omega\}.$$

They proved that this definition is independent of the choice of $\mathcal{C}$ and gave a simpler characterization of the invariant:

Lemma 2.1. *Let A be a group with partial decomposition basis $\mathcal{C}$, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Then $\hat{w}_\alpha(c, p, e, A)$ is the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $U(x_i) \sim_\alpha c$ and $U_p(x_i) \sim_\alpha e$ for all $i = 1, \ldots, n$. If no such n exists, $\hat{w}_\alpha(c, p, e, A) = \omega$.*

3. Model theory background

Throughout this paper, we will consider the language $L_{\infty\omega}$ which is an extension of an ordinary first order language L that allows infinite conjunctions and disjunctions. In this paper we will take L to be the language of group theory with 0, $+$ and $-$. Each formula φ in this language has a quantifier rank $qr(\varphi)$, an ordinal that represents how deeply nested the quantifiers are. The collection of all formulas of quantifier rank less than or equal to α is called $L^\alpha_{\infty\omega}$. See Barwise and Eklof [**BE**] for details. We will write $A \equiv_\infty B$ to represent equivalence in $L_{\infty\omega}$ and $A \equiv_\alpha B$ to represent equivalence in $L^\alpha_{\infty\omega}$.

In the process of proving their classification theorem Barwise and Eklof proved the following expressibility lemma for torsion groups, which is of interest in its own right [**BE**]. It says that p-height is expressible in $L_{\infty\omega}$ and gives the quantifier rank of the formula. The proof applies just as well to arbitrary groups.

Lemma 3.1. *Let $\alpha = \omega\delta + n$ be an ordinal where $n < \omega$. Then there is an existential formula $\theta_\alpha(x)$ whose quantifier rank is*

$$\delta \text{ if } n = 0,$$
$$\delta + 1 \text{ if } n > 0$$

such that for any group A and any $a \in A$, $A \models \theta_\alpha[a]$ if and only if $a \in p^\alpha A$.

4. The classification theorem

Let $\mathbb{Z}_p$ denote the ring of integers localized at the prime p. Jacoby, Leistner, Loth and Strüngmann [**JLLS**] proved that the modified Ulm-Kaplansky and Warfield invariants classify all groups with partial decomposition bases in $L^\delta_{\infty\omega}$ in the following sense:

Theorem 4.1. *Let A and B be groups with partial decomposition bases and let δ be an ordinal. Suppose*

(1) $\hat{u}_p(\alpha, A) = \hat{u}_p(\alpha, B)$ *for all primes* p *and* $\alpha < \omega\delta$;

(2) $\hat{w}_{\omega(\nu+1)}(c,p,e,A) = \hat{w}_{\omega(\nu+1)}(c,p,e,B)$ *for every compatibility class* c *of Ulm matrices, prime* p*, equivalence class* e *of Ulm sequences and* $\nu < \delta$;

(3) *if* $l(t(A \otimes \mathbb{Z}_p)) < \omega\delta$*, then* $\hat{u}_p(\infty, A) = \hat{u}_p(\infty, B)$ *for all primes* p.

Then $A \equiv_\delta B$.

In this paper we seek to prove the converse of this theorem, specifically, if $A \equiv_\delta B$ for appropriate δ then $\hat{w}_{\omega(\nu+1)}(c,p,e,A) = \hat{w}_{\omega(\nu+1)}(c,p,e,B)$ for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences and $\nu < \delta$.

Notice that $A \equiv_\delta B$ implies (1) and (3) whenever δ is a limit ordinal (cf. [**BE**, Theorem 3.1]). We will need to be able to express the modified Warfield invariant in $L^\delta_{\infty\omega}$. As it turns out, partial decomposition bases are not expressible in $L_{\infty\omega}$ [**J2**], so we turn to a modification of an invariant defined by Stanton [**St**], which we then prove is expressible and equivalent to the definition above. The following are definitions of Stanton [**St**], modified to consider only ordinals up to α. Let α be an ordinal, A a group, $M = [m_{q,i}]$ an Ulm matrix, p a prime and $\mu = M_p$.

$M_\alpha(A) = \{x \in A : U(x) \geq_\alpha M\}$.
$M^*_\alpha(A) = \langle x \in M_\alpha(A) : U(x) \nsim_\alpha M \text{ or } x \text{ is torsion}\rangle$.
$\mu_\alpha A = \{x \in A : U_p(x) \geq_\alpha M_p\}$.
$\mu^*_\alpha A = \langle x \in \mu_\alpha A : |p^i x|_p \neq_\alpha m_{p,i} \text{ for infinitely many } i \text{ or } x \text{ is torsion}\rangle$.
$(M;p)^*_\alpha A = (M^*_\alpha(A) + \mu^*_\alpha A) \cap M_\alpha(A)$.

Let c be a compatibility class of Ulm matrices, p a prime, and e an equivalence class of Ulm sequences. We define the modified Stanton invariant as follows:

$$\widehat{ST}_\alpha(c,p,e,A) = \min\{\sup\{rank(M_\alpha(A)/(M;p)^*_\alpha A)\}, \omega\}$$

where the supremum is defined over all $M \in c$ with $M_p \in e$.

Lemma 4.2. *Let* A *be a group,* α *an ordinal,* p *a prime,* $M = [m_{q,i}]$ *an Ulm matrix and* $M' = pM$.

(1) *Suppose* $m_{p,0} < \alpha$. *Then* $x \mapsto px$ *defines a surjective map* $M_\alpha(A) \to M'_\alpha(A)$. *If* $x \in M_\alpha(A)$*, then* $x \in (M;p)^*_\alpha A$ *if and only if* $px \in (M';p)^*_\alpha A$. *Hence* $x \mapsto px$ *induces an isomorphism*

$$M_\alpha(A)/(M;p)^*_\alpha A \xrightarrow{\sim} M'_\alpha(A)/(M';p)^*_\alpha A.$$

(2) *If* $m_{p,0} \geq \alpha$ *then* $M_\alpha(A) = M'_\alpha(A)$ *and* $(M;p)^*_\alpha A = (M';p)^*_\alpha A$.

Proof. Letting $M' = [m'_{q,i}]$ we have for every $i < \omega$, $m'_{p,i} = m_{p,i+1}$ and $m'_{q,i} = m_{q,i}$ if $q \neq p$. Let $\mu' = M'_p$.

For case (1), suppose $m_{p,0} < \alpha$. If $x \in M_\alpha(A)$ then $U(x) \geq_\alpha M$, so for any $i < \omega$ we have $|p^i(px)|_p \geq_\alpha m_{p,i+1} = m'_{p,i}$, hence $px \in M'_\alpha(A)$. If $x' \in M'_\alpha(A)$, then $U(x') \geq_\alpha M'$, and in particular $|x'|_p \geq_\alpha m'_{p,0} = m_{p,1} > m_{p,0}$. Then there is an $x \in A$ such that $|x|_p \geq m_{p,0}$ and $px = x'$. It may be verified that $x \in M_\alpha(A)$, hence $x \mapsto px$ maps $M_\alpha(A)$ onto $M'_\alpha(A)$.

Now let $x \in M_\alpha(A)$. First we assume that $x \in (M;p)^*_\alpha A$, say $x = z_1 + \cdots + z_k$ where $z_1, \ldots, z_k$ are generators of $(M;p)^*_\alpha A$. Consider $z = z_i$ for some i. If z is torsion, so is pz. Suppose z is not torsion. If z is a generator of $M^*_\alpha(A)$, then $z \in M_\alpha(A)$ and $U(pz) \sim U(z) \nsim_\alpha M \sim M'$. Since $pz \in M'_\alpha(A)$ we obtain $pz \in M'^*_\alpha(A)$.

Now suppose z is a generator of $\mu^*_\alpha A$. Then $|p^i pz|_p = |p^{i+1}z|_p \neq_\alpha m_{p,i+1} = m'_{p,i}$ for infinitely many i. In any case, $pz \in M'^*_\alpha(A) + \mu'^*_\alpha A$. Since $z_1 + \ldots + z_k \in M_\alpha(A)$ we have $p(z_1 + \ldots + z_k) \in (M'; p)^*_\alpha A$, hence $px \in (M'; p)^*_\alpha A$.

Conversely, assume that $px \in (M'; p)^*_\alpha A$, say $px = z'_1 + \cdots + z'_k$ for $z'_1, \ldots, z'_k$ generators of $(M'; p)^*_\alpha A$. Let z' be one of the z'_i. Then $U_p(z') \geq_\alpha M'_p$. In particular, $|z'|_p \geq_\alpha m'_{p,0} = m_{p,1} > m_{p,0}$, so there is $z \in A$ such that $pz = z'$ and $|z|_p \geq m_{p,0}$. Also, for $j \geq 1$, $|p^j z|_p = |p^{j-1}z'|_p \geq_\alpha m'_{p,j-1} = m_{p,j}$, so $z \in \mu_\alpha A$. If z' is torsion, so is z. Now suppose z' is not torsion. If z' is a generator of $\mu'^*_\alpha A$ then $|p^i z'|_p \neq_\alpha m'_{p,i}$ for infinitely many i, so $|p^{i+1}z|_p = |p^i z'|_p \neq_\alpha m'_{p,i} = m_{p,i+1}$ for infinitely many i and $z \in \mu^*_\alpha A$ follows. Now suppose z' is a generator of $M'^*_\alpha(A)$. Then there is $z \in M_\alpha(A)$ such that $z' = pz$ and we have $U(z) \sim U(z') \nsim_\alpha M' \sim M$, hence $z \in M^*_\alpha(A)$. In all cases we obtain $z \in M^*_\alpha(A) + \mu^*_\alpha A$. Let $y = z_1 + \ldots + z_k$. Then $y \in M^*_\alpha(A) + \mu^*_\alpha A$ and $py = z'_1 + \ldots + z'_k = px$. Since $U_p(y) \geq_\alpha M_p$ and $|q^i y|_q = |q^i py|_q = |q^i px|_q \geq_\alpha m_{q,i}$ for all $i < \omega$ and $q \neq p$ we have $y \in M_\alpha(A)$, hence $y \in (M; p)^*_\alpha A$. Since $x - y \in t(M_\alpha(A))$ we conclude that $x = y + (x-y) \in (M; p)^*_\alpha A$.

For case (2), suppose $m_{p,0} \geq \alpha$. Then $M =_\alpha M'$ and a straightforward computation yields $M_\alpha(A) = M'_\alpha(A)$, $M^*_\alpha(A) = M'^*_\alpha(A)$ and $\mu^*_\alpha A = \mu'^*_\alpha A$. □

Applying Lemma 4.2 successively allows us to substitute the Ulm matrix M by a p-multiple when calculating the Stanton invariant:

Corollary 4.3. *Let A be a group, α an ordinal, p a prime, M an Ulm matrix and $n < \omega$. Then $M_\alpha(A)/(M; p)^*_\alpha A \cong (p^n M)_\alpha(A)/(p^n M; p)^*_\alpha A$.*

Lemma 4.4. *Let A be a group with partial decomposition basis $\mathcal{C}$, α an ordinal, p a prime, c a compatibility class of Ulm matices and e an equivalence class of Ulm sequences. If $M \in c$ and $M_p \in e$, then rank$(M_\alpha(A)/(M; p)^*_\alpha A) =_\omega \hat{w}_\alpha(c, p, e, A)$.*

Proof. By a suitable choice of $\mathcal{C}$ we may assume that if $X \in \mathcal{C}$ then for any $a_1, \ldots, a_n \in \mathbb{Z}\backslash\{0\}$ and $x_1, \ldots, x_n \in X$, $\{a_1x_1, \ldots, a_nx_n\} \in \mathcal{C}$. Let $M = [m_{q,i}] \in c$ such that $M_p \in e$. Suppose $\hat{w}_\alpha(c, p, e, A) \geq n$, say $x_1, \ldots, x_n \in X$, $X \in \mathcal{C}$ and $U(x_i) \sim_\alpha c$, $U_p(x_i) \sim_\alpha e$ for $1 \leq i \leq n$ (cf. Lemma 2.1).

Consider x_i for some $1 \leq i \leq n$. Since $U(x_i) \sim_\alpha c$ and $M \in c$, there is an m_i such that $m_i U(x_i) \geq_\alpha M$. Then for $m = \prod_{i=1}^n m_i$ we have $U(mx_i) \geq_\alpha M$. Since $U_p(mx_i) \sim U_p(x_i) \sim_\alpha M_p$, there are k_i and k'_i such that $U_p(p^{k_i} mx_i) =_\alpha p^{k'_i} M_p$. Letting $k' = \sum_{i=1}^n k'_i$ and $l_i = k_i + \sum_{j \neq i} k'_j$ for each i, we have $U_p(p^{l_i} mx_i) =_\alpha p^{k'} M_p$. Note that for $q \neq p$, $U_q(p^{l_i} mx_i) = U_q(mx_i) \geq_\alpha M_q = (p^{k'}M)_q$, so $U(p^{l_i} mx_i) \geq_\alpha p^{k'} M$. Writing x_i instead of $p^{l_i} mx_i$, as the choice of $\mathcal{C}$ allows us to do, and writing M instead of $p^{k'} M$, as the corollary allows us to do, we see that we may assume $x_i \in M_\alpha(A)$ and $U_p(x_i) =_\alpha M_p$ for all i.

We wish to prove that $x_1, \ldots, x_n$ are independent representatives of $M_\alpha(A)$ over $(M; p)^*_\alpha A$ and of either prime or infinite order. Suppose

$$a_1x_1 + \cdots + a_nx_n = z_1 + \cdots + z_r \in M_\alpha(A)$$

where $Z = \{z_1, \ldots, z_r\}$ is a set of generators of $\mu^*_\alpha A$ and $M^*_\alpha(A)$. Let $Y \in \mathcal{C}$, where $X \subseteq Y$ and $Z \subseteq \langle Y \rangle^0$. There is a multiplier $p^k a$, where a is relatively prime to p, such that $p^k aZ \subseteq \langle Y \rangle$.

Consider $z \in Z$. First suppose z is not torsion. Suppose z is a generator of $M^*_\alpha(A)$. Then $z \in M_\alpha(A)$ and $U(z) \nsim_\alpha M$. Let $p^k az = c_1x_1 + \cdots + c_nx_n + y$ for some $y \in \langle Y \setminus \{x_1, \ldots, x_n\} \rangle$. For every $m > 0$ there are q and j such that

$|q^j z|_q > m_{q,j+|m|_q}$ and $m_{q,j+|m|_q} < \alpha$. Take arbitrary m and associated q and j and get $m_{q,j+|m|_q} < |q^j p^k az|_q = \min\{|q^j c_i x_i|_q, |q^j y|_q\} \le |q^j c_i x_i|_q$ for all i. Then $m_{q,j+|m|_q} < \alpha$ and either $|q^j c_i x_i|_q \ge \alpha$ or else $m_{q,j+|m|_q} < |q^j c_i x_i|_q < \alpha$. This means that $U(c_i x_i) \nsim_\alpha M$. If $c_i \ne 0$, this would contradict $U(c_i x_i) \sim U(x_i) \sim_\alpha M$, so we must have $c_i = 0$ for all i and so $p^k az \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$.

Now let $z \in Z$ be a generator of $\mu_\alpha^* A$. Given any $j_0 < \omega$ we can find j such that $j + k \ge j_0$ and $|p^{j+k} z|_p >_\alpha m_{p,j+k}$ therefore $m_{p,j+k} < \alpha$. We may write $p^k az = b_1 x_1 + \cdots + b_n x_n + y$ for some $y \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Then for all i, $|p^{j+k} x_i|_p = m_{p,j+k} < |p^j p^k az|_p \le |p^j b_i x_i|_p$, since the x_i form a decomposition set. From this it follows that $p^{k+1} | b_i$ for all i. Now we may write $p^k az = p^{k+1}x + y$, for some $x \in \langle x_1, \ldots, x_n\rangle$ and $y \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Finally, if z is torsion, $p^k az = 0$ is also in this form.

In either case it follows that $p^k a(a_1 x_1 + \cdots + a_n x_n)$ may be written in the form $p^{k+1}x + y$ for some $x \in \langle x_1, \ldots, x_n\rangle$ and $y \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Then for any j, $p^j p^k a(a_1 x_1 + \cdots + a_n x_n) = p^{j+k+1}x + p^j y$. Equating x_i terms, $p^{j+k} a a_i x_i$ is a multiple of $p^{j+k+1} x_i$ for all i.

First suppose $m_{p,j} < \alpha$ for all j and consider $1 \le i \le n$. From this we see $|p^{j+k} a a_i x_i|_p > |p^{j+k} x_i|_p = m_{p,j+k}$ for all j and so $a a_i x_i \in \mu_\alpha^* A$. Since a is relatively prime to p, $a_i x_i \in \mu_\alpha^* A$. Also $a_i x_i \in M_\alpha(A)$, so $a_i x_i \in (M;p)_\alpha^* A$ for all i, proving independence. For each i $px_i \in \mu_\alpha^* A \cap M_\alpha(A)$. Also $x_i \notin (M;p)_\alpha^* A$ since otherwise $p^k a x_i$ would be a multiple of $p^{k+1} x_i$ for some $k \ge 0$ and a relatively prime to p, a contradiction. This proves that each $x_i + (M;p)_\alpha^* A$ has order p. It follows that $rank(M_\alpha(A)/(M;p)_\alpha^* A) \ge n$.

Next suppose $m_{p,j} \ge \alpha$ for some j. By taking p-multiples of each x_i and M we may assume $m_{p,j} \ge \alpha$ for all j. Then for any z of $\mu_\alpha A$, $|p^j z|_p \ge \alpha$ and $|p^j z|_p =_\alpha \alpha =_\alpha m_{p,j}$ for all j. It follows that $\mu_\alpha^* A = t(\mu_\alpha A)$. If $z \in Z$ is not torsion, z must be a generator of $M_\alpha^*(A)$ and hence $p^k az \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. If z is torsion, $p^k az = 0 \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Equating the x_i terms gives $p^k a a_i x_i = 0$, so $a_i x_i$ is torsion and hence in $(M;p)_\alpha^* A$ for all i, again proving independence. We claim the order of each $x_i + (M;p)_\alpha^* A$ is infinite. Suppose $m x_i \in (M;p)_\alpha^* A$ for some $m \ne 0$. Then, as before, $p^k a m x_i = 0$, a contradiction since x_i is in a decomposition set. Again it follows that $rank(M_\alpha(A)/(M;p)_\alpha^* A) \ge n$.

Now suppose $\hat{w}_\alpha(c,p,e,A) = n$ and $\{x_1, \ldots, x_n\}$ is a maximal set. Again we may assume that $x_i \in M_\alpha(A)$ and $U_p(x_i) =_\alpha M_p$ for all i. We wish to show that $\{x_1 + (M;p)_\alpha^* A, \ldots, x_n + (M;p)_\alpha^* A\}$ is a maximal independent set in $M_\alpha(A)/(M;p)_\alpha^* A$. Suppose $x_1, \ldots, x_n, y$ are independent representatives. Choose $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $y \in \langle Y\rangle^0$.

We may write $ay = a_1 x_1 + \cdots + a_n x_n + b_1 y_1 + \cdots + b_m y_m$ where $a > 0$ and $y_1, \ldots, y_m \in \langle Y \setminus \{x_1, \ldots, x_n\}\rangle$. Since $y \in M_\alpha(A)$, for all q and k,

$$m_{q,k} \le_\alpha |q^k y|_q \le |q^k ay|_q = \min_{i,j}\{|q^k a_j x_j|_q, |q^k b_i y_i|_q\}$$

and so for all i, $|q^k b_i y_i|_q \ge_\alpha m_{q,k}$ and $b_i y_i \in M_\alpha(A)$. Since $\{x_1, \ldots, x_n, b_i y_i\} \in \mathcal{C}$ for all i by our choice of $\mathcal{C}$, we find that for each i either $U(b_i y_i) \nsim_\alpha c$ or $U_p(b_i y_i) \nsim_\alpha e$, by the maximality of $\{x_1, \ldots, x_n\}$. First suppose $U(b_i y_i) \nsim_\alpha c$. Then $U(b_i y_i) \nsim_\alpha M$ and $b_i y_i \in M_\alpha^*(A)$. Now suppose $U_p(b_i y_i) \nsim_\alpha e$. Suppose for some k_0 $|p^k b_i y_i|_p =_\alpha m_{p,k}$ for all $k \ge k_0$. Then $p^{k_0} U_p(b_i y_i) =_\alpha p^{k_0} M_p$ and $M_p \sim_\alpha U_p(b_i y_i) \nsim_\alpha e \sim M_p$, a contradiction. This leads to $|p^k b_i y_i| \ne_\alpha m_{p,k}$ for infinitely many k and so $b_i y_i \in \mu_\alpha^* A \cap M_\alpha(A)$. In either case $b_i y_i \in (M;p)_\alpha^* A$.

Then $ay + (M;p)^*_\alpha A = a_1x_1 + \cdots + a_nx_n + (M;p)^*_\alpha A$, contradicting independence. This proves that $rank(M_\alpha(A)/(M;p)^*_\alpha A) =_\omega \hat{w}_\alpha(c,p,e,A)$ for any $M \in c$ with $M_p \in e$. □

Corollary 4.5. *Suppose A is a group with a partial decomposition basis. Then $\widehat{ST}_\alpha(c,p,e,A) = \hat{w}_\alpha(c,p,e,A)$ for all α, c, p and e.*

Corollary 4.6. *Suppose A is a group with a partial decomposition basis. Then $\widehat{ST}(c,p,e,A) = \hat{w}(c,p,e,A)$ for all c, p and e.*

Proof. Choose $\alpha = \sup\{p\text{-}length(A) : p \text{ prime}\}$ □

Corollary 4.7. *If A is a group with partial decomposition basis, then*

$$\widehat{ST}_\alpha(c,p,e,A) = \min\{rank(M_\alpha(A)/(M;p)^*_\alpha A), \omega\}$$

for every $M \in c$ with $M_p \in e$.

Lemma 4.8. *For any $\alpha = \omega\delta + n$ where $n < \omega$, "$\widehat{ST}_\alpha(c,p,e,A) \geq m$" is expressible in a formula of quantifier rank $\leq \delta + \omega + m$.*

Proof. We follow the proof of the local case [**GLLS**]. Let $M \in c$ such that $M_p \in e$.

"$x \in M_\alpha(A)$" if and only if for all q, i, either $q^ix \in q^{m_{q,i}}A$ and $m_{q,i} < \alpha$ or $q^ix \in q^\alpha A$. By Lemma 3.1, "$x \in q^\alpha A$" has quantifier rank δ or $\delta+1$, so this formula has quantifier rank $\leq \delta + 1$.

"x is a generator of $M^*_\alpha(A)$" if and only if $x \in M_\alpha(A)$ and either $U(x) \nsim_\alpha M$ or x is torsion. "x is torsion" has quantifier rank 0 and "$x \in M_\alpha(A)$" has quantifier rank $\leq \delta + 1$. For each $x \in M_\alpha(A)$, "$U(x) \nsim_\alpha M$" if and only if $jM \ngeq_\alpha U(x)$ for all j if and only if for every j there are q and i such that $|q^ix| > m_{q,i+|j|_q}$ and $\alpha > m_{q,i+|j|_q}$ if and only if $\bigwedge_j \bigvee_{q,i}(q^ix \in q^{m_{q,i+|j|_q}+1}A \wedge m_{q,i+|j|_q} < \alpha)$. This can be expressed by a formula of quantifier rank $\leq \delta + 1$.

"$x \in \mu_\alpha A$" if and only if for all i either $p^ix \in p^{m_{p,i}}A$ and $m_{p,i} < \alpha$ or $p^ix \in p^\alpha A$, so it can be expressed by a formula of quantifier rank $\leq \delta + 1$.

"x is a generator of $\mu^*_\alpha A$" if and only if $x \in \mu_\alpha A$ and either $|p^ix|_p > m_{p,i}$ for infinitely many i with $m_{p,i} < \alpha$ or x is torsion. This can be expressed by a formula of quantifier rank $\leq \delta + 1$.

"$x \in (M;p)^*_\alpha A$" if and only if $x \in M_\alpha(A)$ and for some $k \in \omega$ $\exists x_1 \ldots \exists x_k$ such that $x = \sum \lambda_jx_j$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}$ and each x_j either is one of the generators of $\mu^*_\alpha A$, or else is one of the generators of $M^*_\alpha(A)$. It follows that "$x \in (M;p)^*_\alpha A$" can be expressed by a formula of quantifier rank $\leq \delta + \omega$.

"$\widehat{ST}_\alpha(c,p,e,A) \geq m$" if and only if $\exists x_1 \ldots \exists x_m$ ($x_1, \ldots, x_m \in M_\alpha(A)$ and $x_1, \ldots, x_m$ are independent elements of infinite and prime power order modulo $(M;p)^*_\alpha A$). Since "independent modulo $(M;p)^*_\alpha A$" has quantifier rank $\leq \delta + \omega$, this formula has quantifier rank $\leq \delta + \omega + m$. □

Theorem 4.9. *Suppose A and B are groups with partial decomposition bases. If $A \equiv_\lambda B$ where $\lambda = \omega\gamma$ for some γ a limit ordinal, then for all $\alpha < \omega\lambda$, $\hat{w}_\alpha(c,p,e,A) = \hat{w}_\alpha(c,p,e,B)$ for all c, p, e.*

Proof. Write $\alpha = \omega\delta + n$ and $\delta = \omega\delta' + n'$. By Corollary 4.5, $\hat{w}_\alpha = \widehat{ST}_\alpha$. By Lemma 4.8, $\widehat{ST}_\alpha(c,p,e,A) \geq m$ may be expressed in a formula $\phi_{p,m,\alpha}$ with quantifier rank $\leq \delta + \omega + m$. Note $\delta' < \gamma$, so since γ is a limit ordinal, $\delta' + 3 < \gamma$,

$\lambda = \omega\gamma > \omega(\delta' + 3) > \omega\delta' + n' + \omega + m = \delta + \omega + m$. Since $A \equiv_\lambda B$, A and B satisfy the same formulas of quantifier rank $\leq \lambda$. In particular, $A \models \phi_{p,m,\alpha}$ if and only if $B \models \phi_{p,m,\alpha}$. In other words, $\widehat{ST}_\alpha(c,p,e,A) \geq m$ if and only if $\widehat{ST}_\alpha(c,p,e,B) \geq m$. Then $\hat{w}_\alpha(c,p,e,A) = m$ if and only if $\widehat{ST}_\alpha(c,p,e,A) = m$ if and only if $\widehat{ST}_\alpha(c,p,e,B) = m$ if and only if $\hat{w}_\alpha(c,p,e,B) = m$. □

Corollary 4.10. *Let A and B be groups with partial decomposition bases such that $A \equiv_\lambda B$ where $\lambda = \omega\gamma$ and γ is a limit ordinal. Then*

(1) *$\hat{u}_p(\alpha, A) = \hat{u}_p(\alpha, B)$ for all primes p and $\alpha < \omega\lambda$;*
(2) *$\hat{w}_{\omega(\nu+1)}(c,p,e,A) = \hat{w}_{\omega(\nu+1)}(c,p,e,B)$ for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences and $\nu < \lambda$;*
(3) *if $l(t(A \otimes \mathbb{Z}_p)) < \omega\lambda$, then $\hat{u}_p(\infty, A) = \hat{u}_p(\infty, B)$.*

Proof. (1) and (3) follow from [**BE**, Theorem 3.1], and (2) follows from Theorem 4.9. □

Corollary 4.11. *Suppose A and B are groups with partial decomposition bases. If $A \equiv_\infty B$ then $\hat{w}(c,p,e,A) = \hat{w}(c,p,e,B)$ for all c, p, e.*

References

[BE] J. Barwise and P. C. Eklof, *Infinitary properties of abelian torsion groups*, Ann. Math. Logic **2** (1970), 25-68. MR0279180 (43:4906)

[F1] L. Fuchs, *Infinite Abelian Groups*, Vols. **I** and **II**, Academic Press, New York (1970 and 1973). MR0255673 (41:333)

[F2] L. Fuchs, *Abelian p-Groups and Mixed Groups*, University of Montreal Press (1980). MR569744 (82f:20081)

[GLLS] R. Göbel, K. Leistner, P. Loth, L. Strüngmann, *Infinitary equivalence of $\mathbb{Z}_p$-modules with nice decomposition bases*, J. Comm. Algebra **3** (2011), 321-348. MR2826477

[H] P. Hill, *On the classification of abelian groups*, 1967 (unpublished).

[HR] R. Hunter and F. Richman, *Global Warfield groups*, Trans. Amer. Math. Soc. **266** (1981), 555-572. MR617551 (82k:20091)

[J1] C. Jacoby, *Abelian groups with partial decomposition bases in $L_{\infty\omega}$*, in preparation.

[J2] C. Jacoby, *The classification in $L_{\infty\omega}$ of groups with partial decomposition bases*, PhD thesis 1980, University of California, Irvine, revised version (2010).

[JLLS] C. Jacoby, K. Leistner, P. Loth, L. Strüngmann, *Abelian groups with partial decomposition bases in $L^\delta_{\infty\omega}$, Part I*, Contemp. Math., Amer. Math. Soc., Providence, RI (accepted for publication).

[L] P. Loth, *Classifications of Abelian Groups and Pontrjagin Duality*, Algebra, Logic and Application Series, Volume 10, Gordon and Breach Science Publishers, Amsterdam (1998). MR1741074 (2001c:20117)

[R] P. Rothmaler, *Introduction to Model Theory*. Prepared by Frank Reitmaier, translated and revised from the 1995 German original by the author. Algebra, Logic and Application Series, Volume 15, Gordon and Breach Science Publishers, Amsterdam (2000). MR1800596 (2001h:03002)

[St] R. Stanton, *Decomposition bases and Ulm's theorem*, Lecture Notes in Math., Vol. 616, Springer Verlag, New York (1976), pp. 39-56. MR0498902 (58:16913)

[Sz] W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. **41** (1954), 203-271. MR0072131 (17:233e)

[U] H. Ulm, *Zur Theorie der abzählbar-unendlichen abelschen Gruppen*, Math. Ann. **107** (1933), 774-803. MR1512826

[Wal] E. A. Walker, *Ulm's theorem for totally projective groups*, Proc. Amer. Math. Soc. **37** (1973), 387-392 MR0311805 (47:367)

[War] R. Warfield, *Classification theory of abelian groups II: Local theory*, Lecture Notes in Math., Vol. 874, Springer Verlag, New York (1981), pp. 322-349. MR645938 (83h:20052)

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