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Musical Sound: A Mathematical Approach to Timbre

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Section 1 – Introduction

Music and mathematics both play an integral part in human daily life. To a certain extent, one may wonder to what extent mathematics and music are related. More specifically, this paper will aim to explain this relationship through answering another question: What is the mathematical reasoning behind the ear's ability to distinguish two completely different musical sounds? In answering this question, one must call to mind a fundamental term with regards to music: *timbre*. The Oxford English Dictionary defines timbre as "the character or quality of a sound...depending upon the particular voice or instrument producing it…caused by the proportion in which the fundamental tone is combined with the harmonics or overtones" (OED, 2015). Simply put, timbre is what makes one sound different from another sound. Ultimately, the mathematical principles which the concept of timbre will depend on all begin with a closer inspection of sine waves and, consequently, the phenomenon that any given sound wave can be broken down into a series of simple sine waves.

Section 2 – Background

The viewpoint of music as a science is certainly not one which is unique to the past month, year, or even century. In fact, Fauvel, Flood, and Wilson, in their text "Music and Mathematics: From Pythagoras to Fractals" suggest that "the conceptual problems involved in the division of musical space were among the most important challenges faced by seventeenth-century mathematicians" (Fauvel, Flood, & Wilson, 2003). This would insinuate that for centuries, at the very least, mathematics and music have been intimately intertwined. From harmony and number theory, to musical scales and group theory, and far beyond, mathematics and music have always had overlaps and connections to explore (Benson, 2007). Most importantly with regards to this paper, the developments of modern acoustical science in the seventeenth century gave rise to a new science of sound – one which explored the very origins of sound itself (Fauvel, Flood, & Wilson, 2003). The science of timbre depends heavily upon these past developments.

It is important for the reader first to understand the difference between "sound" and "musical sound." While sound (in this case, "noise") encompasses anything able to be heard by the human ear, musical sound is much more restrictive. It consists of typically simple wave patterns which are pleasing to the ear, while sound, or "noise," is typically rather chaotic and unpredictable, and may often be displeasing for the ear. See Figures 1 and 2 below, scaled equivalently to depict this difference.

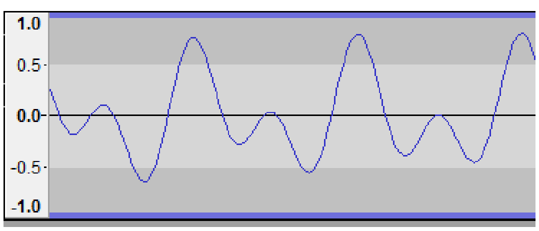

Figure 1. Typical telephone dial tone.

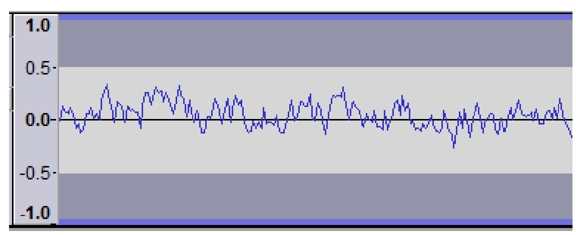

Figure 2. Pink noise.

While the dial tone from Figure 1, which represents musical sound, is rather simplistic with explicitly visible curves, the "pink noise" from Figure 2 is far more complex, with many more peaks and concavities. One would easily deduce that it would likely be far more challenging to generate this "noise" mathematically, compared to how simple it would be to construct a dial tone.

A familiar example of this difference would be the comparison between a note played on a piano and a sound from radio feedback. While the note played on the piano is simple and softsounding to the ear, the sound of radio feedback is rather complex in its waveforms and also rather harsh-sounding to the ear. This paper will seek to solely model musical sound through a mathematical approach.

Section 3 – The Significance of the Sine Wave

It is widely accepted that the sine wave forms the basis of all musical sound. But why has the sine wave been seemingly arbitrarily chosen as this basis? Note that all sound is produced by particles in motion. Now, consider a particle of mass m, vibrating subject to an outside force F towards equilibrium at position $y = 0$. The magnitude of this outside force is directly proportional to the distance y from the equilibrium position. This consideration grants the formula:

$$F = -ky, \text{ where } k \text{ is a constant of proportionality.} \tag{3.1}$$

Recall Newton's laws of motion which give rise to the formula:

$$F = ma, \text{ where } a = \frac{d^2y}{dt^2}; \tag{3.2}$$

i.e. acceleration is the second derivative of the position function y, and t symbolizes time. When combining equations 3.1 and 3.2, one may attain a second-order differential equation, namely:

$$m\left(\frac{d^2y}{dt^2}\right) = -ky \quad \rightarrow \quad \left(\frac{d^2y}{dt^2}\right) + \frac{ky}{m} = 0. \tag{3.3}$$

By solving this differential equation, one may attain the solution y as the functions:

$$y = A\cos(\sqrt{k/m}\, t) + B\sin(\sqrt{k/m}\, t). \tag{3.4}$$

Taking into account that A and B symbolize the initial position and velocity of a particle, let $A = c \sin(\phi)$ and $B = c \cos(\phi)$. Now, by observing the trigonometric identity:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

one may simplify Equation 3.4 by substituting in for A and B to attain the result:

$$y = c \sin\left(\sqrt{k/m}\, t + \phi\right). \tag{3.5}$$

This analysis of particles in motion explains why the sine wave is universally accepted to be the basis of all sound. While it is true that cosine waves and other waves could be used, the sine wave will prove to be the most simplistic and manageable. Now that this has been established, one may examine how any musical sound wave can be constructed through a series of sine waves. (Benson, 2007).

Section 4 – Fourier Theory

Return to the original question presented: what is the mathematical reasoning behind the ear's ability to distinguish two completely different sounds? As explained in Section 3, the sine wave is the basis of all musical sound. Since all sound is produced by different waveforms, it is implicit that all waveforms must be able to be constructed from sine waves. How, then, are these waveforms created? In order to begin answering this question, this section provides an introduction to the concept of Fourier Series.

First, observe that both the cosine function and the sine function are periodic with a period of 2π. In other words, they satisfy:

$$cos(\theta + 2\pi) = cos(\theta) \ ;$$

$$sin(\theta + 2\pi) = sin(\theta) \ .$$

We can then consider any function f which satisfies $f(\theta+2\pi) = f(\theta)$ as a function with period 2π. Jean Baptiste Joseph Fourier introduced the idea that periodic functions can be written as trigonometric series (Benson, 2007). More specifically, he claimed that any function f with period 2π can be rewritten as a trigonometric series, namely:

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(n\theta) + b_n sin(n\theta) \tag{4.1}$$

for constants a_n and b_n. The question then arises: how must one find these constants? (Benson, 2007). Observe the formulae, for integers *m* and *n* such that $m \geq 0$ and $n \geq 0$:

$$\int_0^{2\pi} cos(m\theta)sin(n\theta) = 0 \quad ; \tag{4.2}$$

$$\int_0^{2\pi} cos(m\theta)cos(n\theta)d\theta = \begin{cases} 2\pi & m = n = 0 \\ \pi & m = n > 0 \\ 0 & otherwise \end{cases} . \tag{4.3}$$

$$\int_0^{2\pi} sin(m\theta)sin(n\theta) = \begin{cases} \pi & m = n > 0 \\ 0 & otherwise \end{cases} . \tag{4.4}$$

These formulae can be verified using simple rules of integration, and they will be very useful in helping to solve for a_n and b_n. Now, in order to first solve for a_n, multiply both sides of Equation 4.1 by $cos(m\theta)$ and integrate both sides of the equation, using Equations 4.2 – 4.4 to simplify (Benson, 2007). Observe:

$$\int_0^{2\pi} cos(m\theta)f(\theta)d\theta = \int_0^{2\pi} cos(m\theta)\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n cos(n\theta) + b_n sin(n\theta)\right)\right]d\theta . \tag{4.5}$$

Now, we simplify. A standard theorem of analysis states that, so long as a sum is uniformly continuous, an integral can be passed through an infinite sum (Benson, 2007). Observe the Equations 4.2 – 4.4 and use to simplify the expression to find a_m. We attain:

$$\int_0^{2\pi} cos(m\theta)f(\theta)d\theta = \int_0^{2\pi} \frac{1}{2}a_0 cos(m\theta)d\theta + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} cos(m\theta)cos(n\theta)d\theta + b_n \int_0^{2\pi} cos(m\theta)sin(n\theta)d\theta\right]$$

$$= \frac{a_0}{2m}\left[sin(2m\theta) - sin(0)\right] + [0 + 0 + \ldots + \pi a_m + 0 + \ldots] + [0]$$

$$= \pi a_m . \tag{4.6}$$

Thus, we attain the formula for the coefficient a_m:

$$a_m = \frac{1}{\pi}\int_0^{2\pi} cos(m\theta)f(\theta)d\theta \quad . \tag{4.7}$$

Now, in order to fully apply Fourier Theory, we must find the coefficient b_n. In a similar fashion to the above methods, multiply Equation 4.1 by *sin(mθ)* and integrate. Observe:

$$\int_0^{2\pi} sin(m\theta)f(\theta)d\theta = \int_0^{2\pi} sin(m\theta)\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n cos(n\theta) + b_n sin(n\theta)\right)\right] . \tag{4.8}$$

Similarly, we can distribute the above integral into the above infinite sum, and simplify using Equations 4.2 – 4.4, to attain:

$$\begin{aligned}\int_0^{2\pi} sin(m\theta)f(\theta)d\theta &= \int_0^{2\pi}\frac{1}{2}a_0 sin(m\theta)d\theta + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi} sin(m\theta)cos(n\theta)d\theta + b_n\int_0^{2\pi} sin(m\theta)sin(n\theta)d\theta\right]\\ &= \frac{a_0}{2m}\left[cos(2m\pi) - cos(0)\right] + [0] + [0 + 0 + \ldots + \pi b_m + 0 + \ldots]\\ &= \pi b_m \ . \end{aligned} \tag{4.9}$$

Thus, we attain the formula for the coefficient b_m:

$$b_m = \frac{1}{\pi}\int_0^{2\pi} sin(m\theta)f(\theta)d\theta \quad . \tag{4.10}$$

Now that it has been deduced that any 2π-periodic function can be written as a trigonometric series with constant coefficients, one may be able to actually construct various waveforms and explore their differences (Benson, 2007).

Section 5 – The Square Wave: An Application of Fourier Theory

So far, Section 3 discussed why the sine wave, and not any other periodic oscillating function, is considered the basis of all sound. In addition, Section 4 explained how any 2π-periodic function can be written as a trigonometric series with constant coefficients. This section will now begin to explore how some particularly common musical waveforms associated with musical instruments are constructed.

As the initial question assumed, each musical sound is received differently by the human ear. The primary reason for this is due to each sound's unique waveform. For example, the human ear can certainly tell the difference between the sound of a clarinet being played and the sound of a violin being played. But what makes the waveforms of these so different that the human ear perceives them as different sounds? The answer lies in the modeling of these different waveforms.

One may begin this comparison with an examination of the sound of a clarinet. The sound of a clarinet is widely accepted by musicians and mathematicians alike to be approximated by a square wave (Benson, 2007). See Figure 3 below for an image of a square wave.

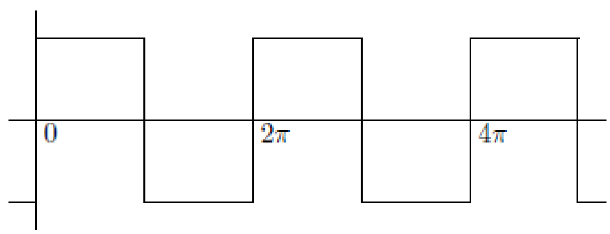

Figure 3. Square Wave. *Music: A Mathematical Offering*. (2007)

If one were to generate a tone with this waveform and compare it to the sound of a tone from a clarinet, one would find them nearly identical in timbre, or tone quality; this would be indicative of the square wave's importance to the sound generated by a clarinet. However, mathematically speaking, this wave is of little relevance because this is clearly not a function – if one were to perform the ever-simple "vertical line test" on this waveform, one would easily deduce that, at any points $n\pi$ on the x-axis (for any integer n), there are multiple corresponding y-values, indicating that this waveform is not a function. However, if this waveform represents the sound produced by a clarinet, and all musical sound waves can be modeled by a series of sine waves, then surely the square wave must be able to be, at the very least, *approximated* by a series of sine waves.

It is here that the previous analysis of Fourier Theory will be of great use. David J. Benson defines the pure square wave as the function $f(\theta)$ defined by:

$$f(\theta) = 1 \text{ (for } 0 \le \theta \le \pi \text{)} \qquad \text{and} \qquad f(\theta) = -1 \text{ (for } \pi \le \theta \le 2\pi \text{).} \tag{5.1}$$

This will account for all values of θ since $f(\theta)$ has period 2π (Benson, 2007). Now, we will solve for the Fourier coefficients for $f(\theta)$ and plug them into the Fourier Series for $f(\theta)$. Observe, by using Equations 4.5 and 4.8 to solve for a_n and b_n:

$$a_m = \frac{1}{\pi}\left[\int_0^\pi cos(m\theta)d\theta + \int_\pi^{2\pi} -cos(m\theta)d\theta\right] = \frac{1}{\pi}\Big[\big(cos(m\pi) - cos(0)\big) - \big(cos(2m\pi) - cos(m\pi)\big)\Big]$$

$$= \frac{1}{\pi}\Big[(0) - (0)\Big] = 0 \, . \tag{5.2}$$

$$b_m = \frac{1}{\pi}\left[\int_0^\pi sin(m\theta)d\theta + \int_\pi^{2\pi} -sin(m\theta)d\theta\right] = \frac{1}{\pi}\left[\left(-\frac{cos(m\theta)}{m} + \frac{cos(0)}{m}\right) + \left(\frac{cos(2m\pi)}{m} - \frac{cos(m\pi)}{m}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^m}{m} + \frac{1}{m} + \frac{1}{m} - \frac{(-1)^m}{m}\right] = \begin{cases} \frac{4}{m\pi}; & m \;\; is \;\; odd \\ 0; & m \;\; is \;\; even \end{cases} . \tag{5.3}$$

Now that we have solved for the coefficients a_n and b_n, let us plug them into our formula for the trigonometric series (Equation 4.1) corresponding to $f(\theta)$ (described in Equation 5.1). We now have, for $a_0 = 0$:

$$f(\theta) = \sum_{n=1}^{\infty}\left[0cos(n\theta) + \frac{4}{n\pi}sin(n\theta)\right] = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}sin(n\theta)$$

$$= \frac{4}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta \ldots\right) \quad . \tag{5.4}$$

Now that the Fourier Series for the square wave has been found, it would be rather simple for one to graph this series as a function to see to what extent the series approximates the shape of the square wave. One may begin by examining various extensions of the series as depicted graphically. Observe the following graphed functions in Figures 4 – 6:

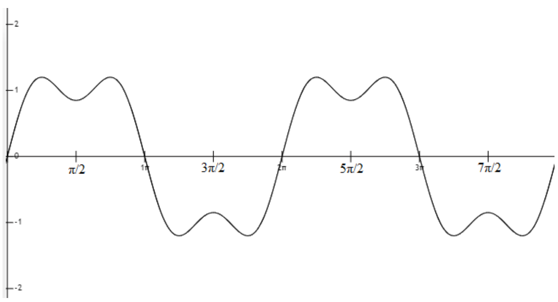

Figure 4. Equation 5.4 taken to $n = 3$.

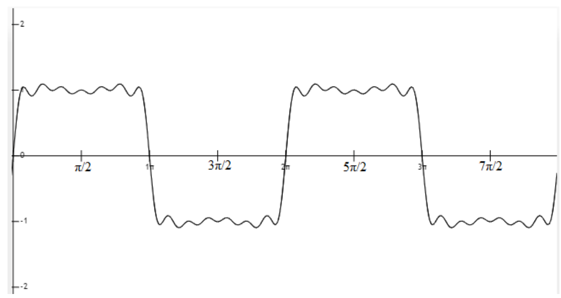

Figure 5. Equation 5.4 taken to $n = 13$.

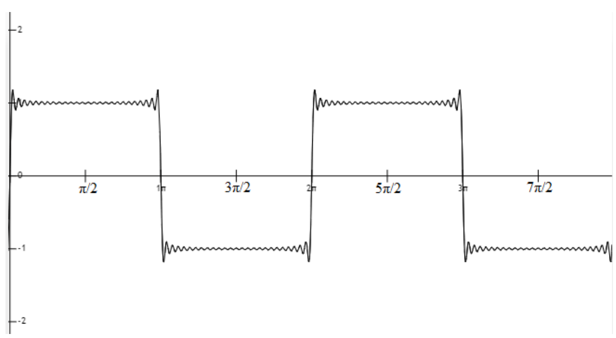

Figure 6. Equation 5.4 taken to $n = 51$.

One may notice that, as n goes to infinity, the Fourier Series more and more closely approximates a square wave (as depicted in Figure 1). Thus, Fourier Series prove to be a very helpful tool in approximating waves which might not fit the definition of a function. As mentioned previously, generating a tone corresponding to this waveform will sound remarkably like a tone produced by a wind instrument such as a clarinet. However – why does the square wave (and consequently the sound of a clarinet) differ mathematically from other waves? The next section will discuss the sawtooth wave, another very common and usable waveform in synthesis, and its musical equivalent – the violin.

<u>Section 6 – The Sawtooth Wave: Fourier Theory Taken Further</u>

In an extension of Sections 4 and 5, one may take Fourier Theory further to examine the waveform associated with another extremely common musical instrument – the violin. Similar to the assumption that the sound of a clarinet is approximated by a square wave, the sawtooth wave has been widely accepted to approximate the sound of a violin (Benson, 2007). See Figure 7 below for an image of a sawtooth wave.

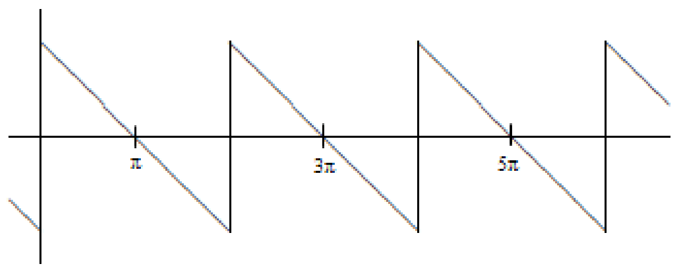

Figure 7. Sawtooth Wave (edited). *Music: A Mathematical Offering*. (2007)

Similar to the issue encountered above with the square wave, one may notice that the pure sawtooth waveform is mathematically insignificant, because this waveform does not represent a function. By performing the "vertical line test" on the waveform, one may observe that, at any point $2n\pi$ on the x-axis (for any integer n), there are multiple corresponding y-values. One may be familiar enough at this point to explore the possibility of approximating the sawtooth waveform using a function generated by a Fourier Series. To begin, observe the 2π-periodic function described by David J. Benson for the pure sawtooth wave:

$$f(\theta) = (\pi - \theta) / 2 \quad (\text{for } 0 \le \theta \le 2\pi) \qquad \text{and} \qquad f(0) = f(2\pi) = 0 . \tag{6.1}$$

We may now use this function to generate the Fourier Series for the sawtooth wave and, in so doing, generate a function to approximate the sawtooth wave (Benson, 2007). Observe, by using Equations 4.5 and 4.8 to solve for the coefficients a_n and b_n:

$$a_m = \frac{1}{\pi}\int_0^{2\pi} cos(m\theta)\left(\frac{\pi-\theta}{2}\right)d\theta = \frac{1}{2\pi}\int_0^{2\pi} cos(m\theta)(\pi-\theta)d\theta \quad . \tag{6.2}$$

Using "u-substitution," we can evaluate this integral as

$$a_m = \left[\frac{(\pi-\theta)sin(m\theta)}{2\pi m}\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi}\frac{-sin(m\theta)}{m}d\theta = \left[\frac{(\pi-\theta)sin(m\theta)}{2\pi m}\right]_0^{2\pi} - \frac{1}{2\pi}\left[\frac{cos(m\theta)}{m^2}\right]_0^{2\pi}$$

$$= 0 - 0 = 0. \tag{6.3}$$

By using "u-substitution" again to evaluate the integral, one may also attain:

$$b_m = \frac{1}{\pi}\int_0^{2\pi} sin(m\theta)\left(\frac{\pi-\theta}{2}\right)d\theta = \left[\frac{-(\pi-\theta)cos(m\theta)}{2\pi m}\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi}\frac{cos(m\theta)}{m}d\theta$$

$$= \left[\frac{-(\pi-\theta)cos(m\theta)}{2\pi m}\right]_0^{2\pi} - \left[\frac{sin(m\theta)}{2\pi m^2}\right]_0^{2\pi} = \frac{1}{m} - 0 = \frac{1}{m}. \tag{6.4}$$

Now that we have attained formulae for any coefficients a_n and b_n, we may evaluate the Fourier Series of the function given by Benson for the sawtooth wave in Equation 6.1. By plugging in for a_n and b_n in the formula given by Equation 4.1, we observe, for $a_0 = 0$:

$$f(\theta) = \sum_{n=1}^{\infty} 0cos(n\theta) + \frac{1}{n}sin(n\theta) = \frac{1}{n}sin(n\theta). \tag{6.5}$$

This function is, therefore, the function which approximates the sawtooth waveform. One may better visualize this by graphing this function and seeing to what extent it approximates the sawtooth waveform given in Figure 5. Observe the following figures:

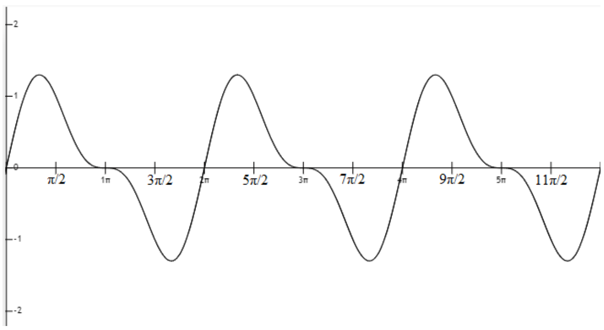

Figure 8. Equation 6.5 taken to n = 2.

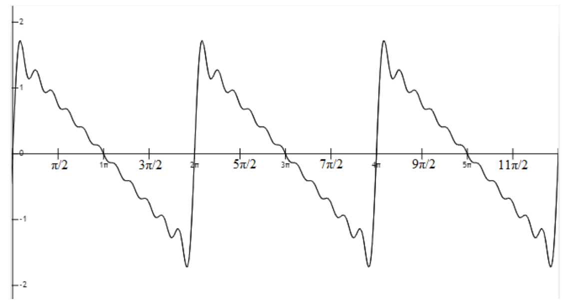

Figure 9. Equation 6.5 taken to n = 11.

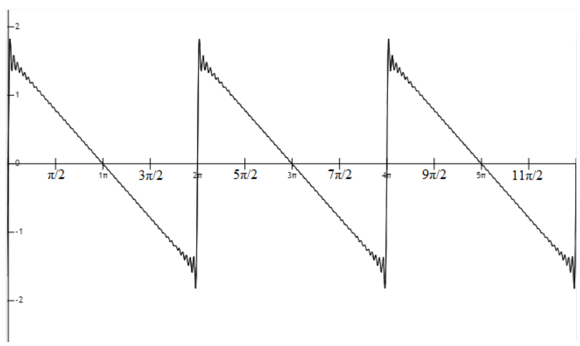

Figure 10. Equation 6.5 taken to n = 51.

By observing the graphs of the function given by Equation 6.5 taken as n goes towards infinity, one may observe that the function approximates the sawtooth waveform more closely as n increases without bound. If one were to generate a tone using this function, it would sound remarkably similar to the tone generated by a bowing of a string on a violin. Now that the mathematical difference between the sounds of a square wave and sawtooth wave (and consequently between a clarinet and violin) has been established, one may wonder: what are the *musical* implications of these Fourier Series on the sounds of the instruments? The following section will connect the previous sections of this paper by summarizing the way that mathematics and music are intertwined through timbre.

Section 7 – Fourier Series and Timbre: An Intimate Connection

Now that it is evident that Fourier Series can be utilized to model musical sound waves, one may pose the question – what musical significance does Fourier Theory have to timbre? First, it is important to note that, when graphing functions of sound waves generated by Fourier Series, the *y*-axis of the graph symbolizes amplitude, while the *x*-axis represents time. Thus volume, or amplitude, is measured on the *y*-axis, and frequency, or cycles per second (Hz), is measured on the *x*-axis. Further, each Fourier-generated function represents *one* tone, or one sound that the human ear receives. Each tone is made up of infinitely many frequencies stacked on top of each other. Each frequency is mathematically represented by a type of sine wave – hence infinitely many sine waves in the waveforms depicted in previous sections.

The lowest frequency is especially important to a tone and, as such, is given a title: the *fundamental*. It is precisely the fundamental of the tone which the ear perceives to be the *pitch* of the tone – or the "highness" or "lowness" of the sound. So, if the lowest frequency is what the human ear perceives to be pitch, then what importance do the tone's remaining infinite frequencies hold? In music and mathematics alike, these remaining frequencies are referred to as *overtones*, or sound waves whose frequency is an integer multiple of that of the fundamental. In terms of Fourier Theory, the fundamental is the pure "$\sin \theta$" portion, while the overtones are the "$\sin 2\theta$," "$\sin 3\theta$," and so on.

It becomes evident, then, why the fundamental is the loudest frequency and is the particular frequency which the ear perceives as pitch – because, in previous examples, the fundamental is the frequency with the largest coefficient. Since the *y*-axis represents volume (amplitude), any coefficient multiplied by a frequency will change the volume of that frequency. Since the fundamental has the largest coefficient (namely, 1), one may conclude why the fundamental is immediately recognized by the ear. Thus, we can conclude that each tone, represented by a Fourier-generated function, is comprised of a fundamental plus infinitely many overtones.

Now, what significance do these overtones have to the *timbre* of a tone? Observe a comparison between Equations 5.4 and 6.5. Notice that Equation 5.4, which represents the Fourier-generated function for the approximation of a square wave, excludes *all* even overtones. On the contrary, observe that Equation 6.5, which represents the Fourier-generated function for the approximation of a sawtooth wave, includes *all* overtones above the fundamental. There is a clear difference, when one generates a sound wave from these two functions, in the timbre of these two waves. While the fundamental of a tone determines its pitch, it is the series of its overtones which determine its timbre. The mathematical reasoning behind timbre, or the human ear's ability to determine two sounds as different, is in the different series of overtones which the tone possesses above the fundamental.

Section 8: Conclusion and Future Research

Both music and mathematics are extraordinarily important to human life. While it may appear as if the two topics have little in common, it is rather the contrary – music and mathematics are intimately intertwined in ways beyond what this paper presents, and it is valuable to explore these connections. For example, perhaps the reader might feel so inclined to see whether one may model non-musical sound by this introduction to modeling musical sound through Fourier Theory. Perhaps, rather, the reader would be interested in the geometric qualities of the square, sawtooth, and other musical waves. For example, do the angles which the waves give rise to have an effect on the timbre of the sound? Does the jump discontinuity have something to do with timbre? There is much to be done beyond these points, and much further research would be required to pursue such topics. Ultimately, Fourier Series provide an incredibly valuable approach to modeling musical sound and, in so doing, clarifying the bond between music and mathematics.

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