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Pure Extensions of Locally Compact Abelian Groups.

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ABSTRACT - In this paper, we study the group Pext(C, A) for locally compact abelian (LCA) groups A and C. Sufficient conditions are established for Pext(C, A) to coincide with the first Ulm subgroup of Ext(C, A). Some structural information on pure injectives in the category of LCA groups is obtained. Letting $\mathfrak{C}$ denote the class of LCA groups which can be written as the topological direct sum of a compactly generated group and a discrete group, we determine the groups G in $\mathfrak{C}$ which are pure injective in the category of LCA groups. Finally we describe those groups G in $\mathfrak{C}$ such that every pure extension of G by a group in $\mathfrak{C}$ splits and obtain a corresponding dual result.

1. Introduction.

In this paper, all considered groups are Hausdorff topological abelian groups and will be written additively. Let $\mathfrak{L}$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. The Pontrjagin dual group of a group G is denoted by $\widehat{G}$ and the annihilator of $S \subseteq G$ in $\widehat{G}$ is denoted by $(\widehat{G}, S)$. A morphism is called *proper* if it is open onto its image, and a short exact sequence

$$0 \rightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \rightarrow 0$$

in $\mathfrak{L}$ is said to be *proper exact* if ϕ and ψ are proper morphisms. In this case, the sequence is called an *extension of A by C (in $\mathfrak{L}$)*, and A may be identified with $\phi(A)$ and C with $B/\phi(A)$. Following Fulp and Griffith [FG1], we let Ext(C, A) denote the (discrete) group of extensions of A by C. The elements represented by pure extensions of A by C form a subgroup of

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Ext(C,A) which is denoted by Pext(C,A). This leads to a functor Pext from $\mathfrak{L} \times \mathfrak{L}$ into the category of discrete abelian groups. The literature shows the importance of the notion of pure extensions (see for instance [F]). The concept of purity in the category of locally compact abelian groups has been studied by several authors (see e.g. [A], [B], [Fu1], [HH], [Kh], [L1], [L2] and [V]). The notion of topological purity is due to Vilenkin [V]: a subgroup H of a group G is called *topologically pure* if $\overline{nH} = H \cap \overline{nG}$ for all positive integers n. The annihilator of a closed pure subgroup of an LCA group is topologically pure (cf. [L2]) but need not be pure in $\widehat{G}$ (see e.g. [A]). As is well known, Pext(C,A) coincides with

$$\text{Ext}(C,A)^1 = \bigcap_{n=1}^{\infty} n\text{Ext}(C,A),$$

the first Ulm subgroup of Ext(C,A), provided that A and C are discrete abelian groups (see [F]). In the category $\mathfrak{L}$, a corresponding result need not hold: for groups A and C in $\mathfrak{L}$, Ext$(C,A)^1$ is a (possibly proper) subgroup of Pext(C,A), and it coincides with Pext(C,A) if (a) A and C are compactly generated, or (b) A and C have no small subgroups (see Theorem 2.4). If G is pure injective in $\mathfrak{L}$, then G has the form $R \oplus T \oplus G'$ where R is a vector group, T is a toral group and G' is a densely divisible topological torsion group. However, the converse need not be true (cf. Theorem 2.7). Let $\mathfrak{C}$ denote the class of LCA groups which can be written as the topological direct sum of a compactly generated group and a discrete group. Then a group in $\mathfrak{C}$ is pure injective in $\mathfrak{L}$ if and only if it is injective in $\mathfrak{L}$ (see Corollary 2.8). Let G be a group in $\mathfrak{C}$. Then every pure extension of G by a group in $\mathfrak{C}$ splits if and only if G has the form $R \oplus T \oplus A \oplus B$ where R is a vector group, T is a toral group, A is a topological direct product of finite cyclic groups and B is a discrete bounded group. Dually, every pure extension of a group in $\mathfrak{C}$ by G splits exactly if G has the form $R \oplus C \oplus D$ where R is a vector group, C is a compact torsion group and D is a discrete direct sum of cyclic groups (see Theorem 2.11).

The additive topological group of real numbers is denoted by $\boldsymbol{R}$, $\boldsymbol{Q}$ is the group of rationals, $\boldsymbol{Z}$ is the group of integers, $\boldsymbol{T}$ is the quotient $\boldsymbol{R}/\boldsymbol{Z}$, $\boldsymbol{Z}(n)$ is the cyclic group of order n and $\boldsymbol{Z}(p^\infty)$ denotes the quasicyclic group. By G_d we mean the group G with the discrete topology, tG is the torsion part of G and bG is the subgroup of all compact elements of G. Throughout this paper the term "isomorphic" is used for "topologically isomorphic", "direct summand" for "topological direct summand" and "direct product" for "topological direct product". We follow the standard notation in [F] and [HR].

2. Pure extensions of LCA groups.

We start with a result on pure extensions involving direct sums and direct products.

THEOREM 2.1. *Let G be in $\mathfrak{L}$ and suppose $\{H_i : i \in I\}$ is a collection of groups in $\mathfrak{L}$. If H_i is discrete for all but finitely many $i \in I$, then*

$$\operatorname{Pext}\left(\bigoplus_{i\in I} H_i, G\right) \cong \prod_{i\in I} \operatorname{Pext}(H_i, G).$$

If H_i is compact for all but finitely many $i \in I$, then

$$\operatorname{Pext}\left(G, \prod_{i\in I} H_i\right) \cong \prod_{i\in I} \operatorname{Pext}(G, H_i).$$

In general, there is no monomorphism

$$\operatorname{Pext}\left(G, \left(\prod_{i\in I} H_i\right)_d\right) \to \prod_{i\in I} \operatorname{Pext}(G, (H_i)_d).$$

PROOF. To prove the first assertion, let $\pi_i : H_i \to \bigoplus H_i$ be the natural injection for each $i \in I$. Then the map $\phi : \operatorname{Ext}(\bigoplus H_i, G) \to \prod \operatorname{Ext}(H_i, G)$ defined by $E \mapsto (E\pi_i)$ is an isomorphism (cf. [FG1] Theorem 2.13), mapping the group $\operatorname{Pext}(\bigoplus H_i, G)$ into $\prod \operatorname{Pext}(H_i, G)$. If the groups H_i and G are stripped of their topology, the corresponding isomorphism maps the group $\operatorname{Pext}(\bigoplus (H_i)_d, G_d)$ onto $\prod \operatorname{Pext}((H_i)_d, G_d)$ (see [F] Theorem 53.7 and p. 231, Exercise 6). Since an extension equivalent to a pure extension is pure, ϕ maps $\operatorname{Pext}(\bigoplus H_i, G)$ onto $\prod \operatorname{Pext}(H_i, G)$, establishing the first statement. The proof of the second assertion is similar. To prove the last statement, let p be a prime and $H = \prod_{n=1}^{\infty} \boldsymbol{Z}(p^n)$, taken discrete. Assume $\operatorname{Ext}(\widehat{\boldsymbol{Q}}, H) = 0$. By [FG2] Corollary 2.10, the sequences

$$\operatorname{Ext}(\widehat{\boldsymbol{Q}}, H) \to \operatorname{Ext}(\widehat{\boldsymbol{Q}}, H/tH) \to 0$$

and

$$0 = \operatorname{Hom}((\boldsymbol{Q}/\boldsymbol{Z})^\wedge, H/tH) \to \operatorname{Ext}(\widehat{\boldsymbol{Z}}, H/tH) \to \operatorname{Ext}(\widehat{\boldsymbol{Q}}, H/tH)$$

are exact, hence [FG1] Proposition 2.17 yields $H/tH \cong \operatorname{Ext}(\widehat{\boldsymbol{Z}}, H/tH) = 0$ which is impossible. Since $\widehat{\boldsymbol{Q}}$ is torsion-free, it follows that $\operatorname{Pext}(\widehat{\boldsymbol{Q}}, H) = \operatorname{Ext}(\widehat{\boldsymbol{Q}}, H) \neq 0$. On the other hand, we have

$$\prod_{n=1}^{\infty} \operatorname{Pext}(\widehat{\boldsymbol{Q}}, \boldsymbol{Z}(p^n)) = \prod_{n=1}^{\infty} \operatorname{Ext}(\widehat{\boldsymbol{Q}}, \boldsymbol{Z}(p^n)) \cong \prod_{n=1}^{\infty} \operatorname{Ext}(\boldsymbol{Z}(p^n), \boldsymbol{Q}) = 0$$

by [FG1] Theorem 2.12 and [F] Theorem 21.1. Note that this example shows that Proposition 6 in [Ful] is incorrect. □

PROPOSITION 2.2. *Suppose* $E_0 : 0 \to A \xrightarrow{\phi} B \to C \to 0$ *is a proper exact sequence in* $\mathfrak{L}$. *Let* $a : A \to A$ *be a proper continuous homomorphism and* a_* *the induced endomorphism on* $\mathrm{Ext}(C, A)$ *given by* $a_*(E) = aE$. *Then* $E_0 \in \mathrm{Im}\, a_*$ *if and only if* $\mathrm{Im}\,\phi/\mathrm{Im}\,\phi a$ *is a direct summand of* $B/\mathrm{Im}\,\phi a$.

PROOF. If $a : A \to A$ is a proper morphism in $\mathfrak{L}$, then

$$0 \to \mathrm{Im}\, a \to A \to \mathrm{Im}\,\phi/\mathrm{Im}\,\phi a \to 0$$

and

$$0 \to \mathrm{Ker}\, a \to A \to \mathrm{Im}\, a \to 0$$

are proper exact sequences in $\mathfrak{L}$ (cf. [HR] Theorem 5.27). Now [FG2] Corollary 2.10 and the proof of [F] Theorem 53.1 show that $E_0 \in \mathrm{Im}\, a_*$ if and only if the induced proper exact sequence

$$0 \to \mathrm{Im}\,\phi/\mathrm{Im}\,\phi a \to B/\mathrm{Im}\,\phi a \to C \to 0$$

splits. □

If A and C are groups in $\mathfrak{L}$, then $\mathrm{Ext}(C,A) \cong \mathrm{Ext}(\widehat{A}, \widehat{C})$ (see [FG1] Theorem 2.12). We have, however:

LEMMA 2.3. *Let* A *and* C *be in* $\mathfrak{L}$. *Then:*

(i) *In general,* $\mathrm{Pext}(C,A) \ncong \mathrm{Pext}(\widehat{A}, \widehat{C})$.

(ii) *Let* $\mathfrak{K}$ *denote a class of LCA groups satisfying the following property: If* $G \in \mathfrak{K}$, *then* $\widehat{G} \in \mathfrak{K}$ *and* nG *is closed in* G *for all positive integers* n. *Then* $\mathrm{Pext}(C,A) \cong \mathrm{Pext}(\widehat{A}, \widehat{C})$ *whenever* A *and* C *are in* $\mathfrak{K}$.

PROOF. (i) The finite torsion part of a group in $\mathfrak{L}$ need not be a direct summand (see for instance [Kh]), so there is a finite group F and a torsion-free group C in $\mathfrak{L}$ such that $\mathrm{Pext}(C, F) = \mathrm{Ext}(C, F) \neq 0$. On the other hand, $\mathrm{Pext}(\widehat{F}, \widehat{C}) \cong \mathrm{Pext}(F, (\widehat{C})_d) = 0$ by [F] Theorem 30.2.

(ii) Let A and C be in $\mathfrak{K}$ and consider the isomorphism $\mathrm{Ext}(C,A) \xrightarrow{\sim} \mathrm{Ext}(\widehat{A}, \widehat{C})$ given by $E : 0 \to A \to B \to C \to 0 \mapsto \widehat{E} : 0 \to \widehat{C} \to \widehat{B} \to \widehat{A} \to 0$. The annihilator of a closed pure subgroup of B is topologically pure in $\widehat{B}$ (cf. [L2] Proposition 2.1) and for all positive integers n, nA and $n\widehat{C}$ are closed subgroups of A and $\widehat{C}$, respectively. Therefore, E is pure if and only if $\widehat{E}$ is pure. □

Recall that a topological group is said to have *no small subgroups* if there is a neighborhood of 0 which contains no nontrivial subgroups. Moskowitz [M] proved that the LCA groups with no small subgroups have the form $\boldsymbol{R}^n \oplus \boldsymbol{T}^m \oplus D$ where n and m are nonnegative integers and D is a discrete group, and that their Pontrjagin duals are precisely the compactly generated LCA groups.

THEOREM 2.4. *For groups A and C in $\mathfrak{L}$, we have:*

(i) $\mathrm{Pext}(C,A) \supseteq \mathrm{Ext}(C,A)^1$.

(ii) $\mathrm{Pext}(C,A) \neq \mathrm{Ext}(C,A)^1$ *in general.*

(iii) *Suppose (a) A and C are compactly generated, or (b) A and C have no small subgroups. Then* $\mathrm{Pext}(C,A) = \mathrm{Ext}(C,A)^1$.

PROOF. (i) Let $a : A \to A$ be the multiplication by a positive integer n and let $E : 0 \to A \xrightarrow{\phi} X \to C \to 0 \in n\mathrm{Ext}(C,A)$. Since Ext is an additive functor, there exists an extension $0 \to A \to B \to C \to 0$ such that

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & a\downarrow & & \downarrow & & \| & & \\ 0 & \to & A & \xrightarrow{\phi} & X & \to & C & \to & 0 \end{array}$$

is a pushout diagram in $\mathfrak{L}$. An easy calculation shows that $nX \cap \phi(A) = n\phi(A)$, hence $\mathrm{Ext}(C,A)^1$ is a subset of $\mathrm{Pext}(C,A)$.

(ii) Let $\mathrm{Pext}(C,F)$ be as in the proof of Lemma 2.3. Then $\mathrm{Pext}(C,F) \neq 0$ but $\mathrm{Ext}(C,F)^1 = 0$.

(iii) Suppose first that A and C are compactly generated. If $a : A \to A$ is the multiplication by a positive integer n, then $a(A) = nA$ is a group in $\mathfrak{L}$. Since A is σ-compact, a is a proper morphism by [HR] Theorem 5.29. Let $E : 0 \to A \xrightarrow{\phi} B \to C \to 0 \in \mathrm{Ext}(C,A)$. By Proposition 2.2, $E \in \mathrm{Im}\, a_* = n\mathrm{Ext}(C,A)$ if and only if $\phi(A)/n\phi(A)$ is a direct summand of $B/n\phi(A)$. Now assume that E is a pure extension. Then $\phi(A)/n\phi(A)$ is pure in the group $B/n\phi(A)$ which is compactly generated (cf. [M] Theorem 2.6). Since the compact group $\phi(A)/n\phi(A)$ is topologically pure, it is a direct summand of $B/n\phi(A)$ (see [L1] Theorem 3.1). Consequently, E is an element of the first Ulm subgroup of $\mathrm{Ext}(C,A)$ and by (i) the assertion follows. To prove the second part of (iii), assume that A and C have no small subgroups. By what we have just shown and Lemma 2.3, we have $\mathrm{Pext}(C,A) \cong \mathrm{Pext}(\widehat{A},\widehat{C}) = \mathrm{Ext}(\widehat{A},\widehat{C})^1 \cong \mathrm{Ext}(C,A)^1$. □

By the structure theorem for locally compact abelian groups, any group G in $\mathfrak{L}$ can be written as $G = V \oplus \widetilde{G}$ where V is a maximal vector subgroup of G and $\widetilde{G}$ contains a compact open subgroup. The groups V and $\widetilde{G}$ are uniquely determined up to isomorphism (see [HR] Theorem 24.30 and [AA] Corollary 1).

LEMMA 2.5. *A group G in $\mathfrak{L}$ is torsion-free if and only if every compact open subgroup of $\widetilde{G}$ is torsion-free.*

PROOF. Only sufficiency needs to be shown. Suppose every compact open subgroup of $\widetilde{G}$ is torsion-free and assume that G is not torsion-free. Then $\widetilde{G}$ contains a nonzero element x of finite order. If K is any compact open subgroup of $\widetilde{G}$, then $K + \langle x \rangle$ is compact (see [HR] Theorem 4.4) and open in $\widetilde{G}$ but not torsion-free, a contradiction. □

Dually, we obtain the following fact which extends [A] (4.33). Recall that a group is said to be *densely divisible* if it possesses a dense divisible subgroup.

LEMMA 2.6. *A group G in $\mathfrak{L}$ is densely divisible if and only if $\widetilde{G}/K$ is divisible for every compact open subgroup K of $\widetilde{G}$.*

PROOF. Again, only sufficiency needs to be proved. Assume that $\widetilde{G}/K$ is divisible for every compact open subgroup K of $\widetilde{G}$ and let C be a compact open subgroup of $(\widehat{G})\widetilde{}$. Since $(\widehat{G})\widetilde{} \cong (G/V)\widehat{}$ where V is a maximal vector subgroup of G, there exists a compact open subgroup X/V of G/V such that $C \cong ((G/V)\widehat{}, X/V) \cong ((G/V)/(X/V))\widehat{}$ (see [HR] Theorems 23.25, 24.10 and 24.11). By our assumption, $(G/V)/(X/V)$ is divisible. But then C is torsion-free (cf. [HR] Theorem 24.23), so by Lemma 2.5, $\widehat{G}$ is torsion-free. Finally, [R] Theorem 5.2 shows that G is densely divisible. □

Let G be in $\mathfrak{L}$. Then G is called *pure injective in* $\mathfrak{L}$ if for every pure extension $0 \to A \xrightarrow{\phi} B \to C \to 0$ in $\mathfrak{L}$ and continuous homomorphism $f : A \to G$ there is a continuous homomorhism $\bar{f} : B \to G$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{\phi} & B & \to & C & \to & 0 \\ & & f \downarrow & \swarrow \bar{f} & & & & & \\ & & G & & & & & & \end{array}$$

is commutative. Following Robertson [R], we call G a *topological torsion group* if $(n!)x \to 0$ for every $x \in G$. Note that a group G in $\mathfrak{L}$ is a topological torsion group if and only if both G and $\widehat{G}$ are totally disconnected (cf. [R] Theorem 3.15). Our next result improves [Ful] Proposition 9.

THEOREM 2.7. *Consider the following conditions for a group G in $\mathfrak{L}$:*

(i) *G is pure injective in $\mathfrak{L}$.*

(ii) $\mathrm{Pext}(X, G) = 0$ *for all groups X in $\mathfrak{L}$.*

(iii) $G \cong \boldsymbol{R}^n \oplus \boldsymbol{T}^{\mathfrak{m}} \oplus G'$ *where n is a nonnegative integer, $\mathfrak{m}$ is a cardinal and G' is a densely divisible topological torsion group which, as such, possesses no nontrivial pure compact open subgroups.*

Then we have: (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *and* (iii) $\not\Rightarrow$ (ii).

PROOF. If G is pure injective in $\mathfrak{L}$, then any pure extension $0 \to G \to B \to X \to 0$ in $\mathfrak{L}$ splits because there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & G & \to & B & \to & X & \to & 0 \\ & & \| & \swarrow & & & & & \\ & & G, & & & & & & \end{array}$$

hence (i) implies (ii). Conversely, assume (ii). If $0 \to A \to B \to X \to 0$ is a pure extension in $\mathfrak{L}$ and $f : A \to G$ is a continuous homomorphism, then there is a pushout diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & X & \to & 0 \\ & & f\downarrow & & \downarrow & & \| & & \\ 0 & \to & G & \to & Y & \to & X & \to & 0. \end{array}$$

The bottom row is an extension in $\mathfrak{L}$ (cf. [FG1]) which is pure. By our assumption, it splits and (i) follows.

To show (ii) $\Rightarrow$ (iii), let us assume first that $\mathrm{Pext}(X, G) = 0$ for all groups $X \in \mathfrak{C}$. Then the proof of [L1] Theorem 4.3 shows that G is isomorphic to $\boldsymbol{R}^n \oplus \boldsymbol{T}^{\mathfrak{m}} \oplus G'$ where n is a nonnegative integer, $\mathfrak{m}$ is a cardinal and G' is totally disconnected. Notice that G'/bG' is discrete (cf. [HR] (9.26)(a)) and torsion-free. Since the sequence

$$0 = \mathrm{Hom}((\boldsymbol{Q}/\boldsymbol{Z})\widehat{\ }, G'/bG') \to \mathrm{Ext}(\widehat{\boldsymbol{Z}}, G'/bG') \to \mathrm{Ext}(\widehat{\boldsymbol{Q}}, G'/bG') = 0$$

is exact, G'/bG' is isomorphic to $\mathrm{Ext}(\widehat{\boldsymbol{Z}}, G'/bG') = 0$ and therefore $G' = bG'$. It follows that the dual group of G' is totally disconnected (cf. [HR] Theorem 24.17), thus G' is a topological torsion group. Suppose that $\mathrm{Pext}(X, G) = 0$ for all $X \in \mathfrak{L}$ and let K be a compact open subgroup of G'. Then G'/K is a divisible group (see [Fu2] Theorem 7 or the proof of [L1] Theorem 4.1), so by Lemma 2.6 G' is densely divisible. Now assume that G' has a pure compact open subgroup A. Since A is algebraically compact, it is a direct summand of G'. But then A is divisible, hence connected (see [HR] Theorem 24.25) and therefore $A = 0$. Consequently, (ii) implies (iii).

Finally, (iii) $\not\Rightarrow$ (ii) because for instance, there is a nonsplitting extension of $\boldsymbol{Z}(p^\infty)$ by a compact group (cf. [A] Example 6.4). □

Those groups in $\mathfrak{C}$ which are pure injective in $\mathfrak{L}$ are completely determined:

COROLLARY 2.8. *A group G in $\mathfrak{C}$ is pure injective in $\mathfrak{L}$ if and only if $G \cong \boldsymbol{R}^n \oplus \boldsymbol{T}^{\mathfrak{m}}$ where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*

PROOF. The assertion follows immediately from [M] Theorem 3.2 and the above theorem. □

The following lemma will be needed.

LEMMA 2.9. *Every finite subset of a reduced torsion group A can be embedded in a finite pure subgroup of A.*

PROOF. By [F] Theorem 8.4, it suffices to assume that A is a reduced p-group. But then the assertion follows from [K] p. 23, Lemma 9 and an easy induction. □

A pure extension $0 \to A \to B \to C \to 0$ with discrete torsion group A and compact group C need not split, as [A] Example 6.4 illustrates. Our next result shows that no such example can occur if A is reduced.

PROPOSITION 2.10. *Suppose A is a discrete reduced torsion group. Then* $\operatorname{Pext}(X, A) = 0$ *for all compactly generated groups X in $\mathfrak{L}$.*

PROOF. Suppose $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} X \to 0$ represents an element of $\operatorname{Pext}(X, A)$ where A is a discrete reduced torsion group and X is a compactly generated group in $\mathfrak{L}$. By [FG2] Theorem 2.1, there is a compactly generated subgroup C of B such that $\psi(C) = X$. If we set $A' = \phi(A)$, then $A' \cap C$ is discrete, compactly generated and torsion, hence finite, so by Lemma 2.9 A' has a finite pure subgroup F containing $A' \cap C$. Now set $C' = C + F$. Then F is a pure subgroup of C' because it is pure in B. But then F is topologically pure in C' since C' is compactly generated. By [L1] Theorem 3.1, there is a closed subgroup Y of C' such that $C' = F \oplus Y$. We have $B = A' + C = A' + C' = A' + Y$ and

$$A' \cap Y = C' \cap A' \cap Y = (F + C) \cap A' \cap Y = [F + (C \cap A')] \cap Y = F \cap Y = 0,$$

thus B is an algebraic direct sum of A' and Y. Since Y is compactly generated, it is σ-compact, so by [FG1] Corollary 3.2 we obtain $B = A' \oplus Y$. Consequently, the extension E splits. □

THEOREM 2.11. *Let G be a group in $\mathfrak{C}$. Then we have:*

(i) $\mathrm{Pext}(X, G) = 0$ *for all* $X \in \mathfrak{C}$ *if and only if* $G \cong \boldsymbol{R}^n \oplus \boldsymbol{T}^{\mathfrak{m}} \oplus A \oplus B$ *where n is a nonnegative integer, $\mathfrak{m}$ is a cardinal, A is a direct product of finite cyclic groups and B is a discrete bounded group.*

(ii) $\mathrm{Pext}(G, X) = 0$ *for all* $X \in \mathfrak{C}$ *if and only if* $G \cong \boldsymbol{R}^n \oplus C \oplus D$ *where n is a nonnegative integer, C is a compact torsion group and D is a discrete direct sum of cyclic groups.*

PROOF. Suppose $G \in \mathfrak{C}$ and $\mathrm{Pext}(X, G) = 0$ for all $X \in \mathfrak{C}$. By the proof of part (ii) $\Rightarrow$ (iii) of Theorem 2.7, G is isomorphic to $\boldsymbol{R}^n \oplus \boldsymbol{T}^{\mathfrak{m}} \oplus A \oplus B$ where A is a compact totally disconnected group and B is a discrete torsion group. By Lemma 2.3, we have $\mathrm{Pext}(\widehat{A}, X) \cong \mathrm{Pext}(\widehat{X}, A) = 0$ for all discrete groups X, hence $\widehat{A}$ is a direct sum of cyclic groups (see [F] Theorem 30.2) and it follows that A is a direct product of finite cyclic groups. Again, we make use of [A] Example 6.4 and conclude that B is reduced. But then B is bounded since it is torsion and cotorsion. Conversely, suppose G has the form $\boldsymbol{R}^n \oplus \boldsymbol{T}^{\mathfrak{m}} \oplus A \oplus B$ as in the theorem and let $X = \boldsymbol{R}^m \oplus Y \oplus Z$ where Y is a compact group and Z is a discrete group. Then $\mathrm{Pext}(X, A) \cong \cong \mathrm{Pext}(\widehat{A}, \widehat{X}) \cong \mathrm{Pext}(\widehat{A}, (\widehat{X})_d) = 0$. By Theorem 2.1, Proposition 2.10 and [F] Theorem 27.5 we have

$$\mathrm{Pext}(X, B) \cong \mathrm{Pext}(\boldsymbol{R}^m, B) \oplus \mathrm{Pext}(Y, B) \oplus \mathrm{Pext}(Z, B) = 0$$

and conclude that

$$\mathrm{Pext}(X, G) \cong \mathrm{Pext}(X, \boldsymbol{R}^n \oplus \boldsymbol{T}^{\mathfrak{m}}) \oplus \mathrm{Pext}(X, A) \oplus \mathrm{Pext}(X, B) = 0.$$

Finally, the second assertion follows from Lemma 2.3 and duality. $\square$

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