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Peter Loth
Sacred Heart University, lothp@sacredheart.edu

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Splitting of the Identity Component in Locally Compact Abelian Groups.

PETER LOTH (*)

Introduction.

Let G be a topological abelian group and H a closed subgroup. If G contains a closed subgroup K such that $H \cap K = \{0\}$, the identity of G, and that the map $(h, k) \to h + k$ is a homeomorphism of $H \times K$ onto G, then G is said to be the *direct sum of H and K*, and we say that *H splits in G*.

For example, the groups splitting in every (discrete) abelian group in which they are contained as subgroups are exactly the divisible groups (cf. Fuchs [3], 24.5), and the LCA (locally compact abelian) groups splitting in every LCA group in which they are contained as closed subgroups are exactly the injective groups in the class of LCA groups, i.e. topologically isomorphic to $\boldsymbol{R}^n \times (\boldsymbol{R}/\boldsymbol{Z})^{\mathfrak{m}}$ for some $n \in N_0$ and cardinal number $\mathfrak{m}$, and likewise these are exactly the connected LCA groups splitting in every LCA group in which they are contained as identity components (compare Ahern-Jewitt [1], Dixmier [2] and Fulp-Griffith [4]). It is well known that this statement remains true if «LCA group» is replaced by «compact abelian group» and groups of the form $(\boldsymbol{R}/\boldsymbol{Z})^{\mathfrak{m}}$ are considered.

In this paper, we are concerned with the splitting of the identity component G_0 in an LCA group G. As Pontrjagin duality shows, this splitting is «dual» to the splitting of the torsion part tA in a discrete abelian group A, if G is assumed to be compact.

An LCA group G contains a splitting identity component G_0, if G/G_0 has closed torsion part; in particular, if G/G_0 is torsion or torsion-free. This result is contained in Proposition 1. Clearly it follows that G_0 splits in G if all compact elements of G/G_0 have finite order, or equiva-

(*) Indirizzo dell'A.: Department of Mathematics, Wesleyan University, Middletown, Connecticut 06459, U.S.A.

lently, if every element of every algebraic complement of G_0 in G generates a closed subgroup of G (Proposition 2). Finally, we will show that G is a topological torsion group modulo G_0 and contains G_0 as a direct summand if and only if there are compact subroups $B^1 \supseteq \ldots \supseteq B^n \supseteq \supseteq \ldots$ of G with trivial intersection such that $G/(G_0 + B^n)$ is torsion and the equality $(B^n)_0 = G_0 \cap B^n$ holds for $n = 1, 2, \ldots$ (Theorem 4).

For details and information on the splitting in discrete abelian groups, and for the results concerning LCA groups and Pontrjagin duality, respectively, we may refer to the books of Fuchs [3] and Hewitt-Ross [5].

Throughout the dual group of G is denoted by $\widehat{G}$, and $(\widehat{G}, H)$ denotes the annihilator of $H \subset G$ in $\widehat{G}$. Of course, all considered groups will be abelian.

As is well known, an LCA group G with closed torsion part tG contains a splitting identity component (cf. Khan [6], p. 523) and since the closedness of tG in G implies $tG_0 = \{0\}$ (see [6], p. 523), we can say a bit more, as the next proposition shows.

PROPOSITION 1. *Consider the following properties for an* LCA *group G:*

(*a*) *stet torsion part of every algebraic complement of G_0 in G is closed in G;*

(*b*) *there exists an algebraic complement U of G_0 in G such that tU is closed in G;*

(*c*) *$tG + G_0$ is closed in G;*

(*d*) *$t(G/G_0)$ is closed in G/G_0;*

(*e*) *G_0 splits in G.*

Then we have:

(i) $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Rightarrow (e)$. *In particular, G_0 splits in G if G/G_0 is torsion or torsion-free.*

(ii) $(e) \not\Rightarrow (d) \not\Rightarrow (a)$ *generally.*

PROOF. $(a) \Rightarrow (b) \Rightarrow (c)$ is obvious. There always exists an algebraic complement of G_0 in G because G_0 is divisible, thus we obtain $(c) \Leftrightarrow (d)$.

Now to the main part. Let $t(G/G_0)$ be a closed subgroup of G/G_0. Assume first that G is compact. By duality, G/G_0 is the direct sum of $t(G/G_0)$ and some closed subgroup F/G_0. Regarding G as dual group of the discrete group $\widehat{G}$, we have $G_0 = (G, t\widehat{G})$ and therefore there exist subgroups B and D of $t\widehat{G}$ with $t(G/G_0) = (G, D)/G_0$ and $F/G_0 = = (G, B)/G_0$. Thus we have $t\widehat{G} = B \oplus D$. Observe that the natural map from G onto $G/(G, B)$ induces a topological isomorphism from $(G/(G, B))^{\frown}$ onto B. Since $G/(G, B) \cong (G, D)/G_0$ is compact and torsion, it is bounded, i.e. B is bounded. On the other hand, D is divisible since its dual groupis torsion-free. Thus $t\hat{G}$ splits in $\hat{G}$ (cf. Fuchs [3], 100.1) and it follows from duality that G_0 splits in G. If G is arbitrary, we may write it as $V \oplus \tilde{G}$, where V is a maximal vector subgroup and $\tilde{G}$ contains a compact open subgroup G'. Since $G'/G_0' = G'/\tilde{G}_0 \cong (V \oplus G')/G_0$ has closed torsion part, $\tilde{G}_0$ splits in G' as we have just shown. Hence, there is a continuous homomorphism f: $G' \to \tilde{G}_0$ such that f is the identity on $\tilde{G}_0$. Since $\tilde{G}_0$ is divisible and G' is open in $\tilde{G}$, we may extend f to a continuous homomorphism $\tilde{f}$: $\tilde{G} \to \tilde{G}_0$, and since $\tilde{f}$ is the identity on $\tilde{G}_0$, $\tilde{G}_0$ splits in $\tilde{G}$. Hence G_0 splits in G, and we get $(d) \Rightarrow (e)$.

Therefore, if (*c*) holds, G is the direct sum of G_0 and some closed subgroup K, where tK is closed in K. Thus we obtain $(c) \Rightarrow (b)$.

The converse of $(d) \Rightarrow (e)$ clearly fails in general, as we see from the compact group $G = \prod_{n \in N} \boldsymbol{Z}/n\boldsymbol{Z}$. Next we consider the compact group $G' = \boldsymbol{R}/\boldsymbol{Z} \times U$, where U is defined as $(\boldsymbol{Z}/n\boldsymbol{Z})^{\aleph_0}$ for some $n \geq 2$. If η is a discontinuous map from U to $\boldsymbol{R}/\boldsymbol{Z}$, the diagonal subgroup $\{(\eta(x), x): x \in U\}$ is a nonclosed algebraic complement of G_0' in G'. Thus (*b*) does not imply (*a*). This finishes the proof.

It follows immediately from Proposition 1, that G_0 splits in an LCA group G if all compact elements of G/G_0 have finite order. An equivalent condition is given in the next proposition.

PROPOSITION 2. *Let G be an* LCA *group. Then the following are equivalent*:

(i) *all compact elements of G/G_0 have finite order*;

(ii) *every element of every algebraic complement of G_0 in G generates a closed subgroup of G.*

PROOF. Suppose that (i) holds and let x be an arbitrary element of any algebraic complement of G_0 in G. Since monothetic LCA groups are compact or topologically isomorphic to $\boldsymbol{Z}$, the factor groups $\overline{\langle x + G_0 \rangle}$ and $\langle x + G_0 \rangle$ are identical. If $\overline{\langle x \rangle}$ is compact, then $\langle x + G_0 \rangle$ is finite, thus $x \in tG$. To prove the converse we may write G as $V \oplus \tilde{G}$, where V is a maximal vector subgroup and $\tilde{G}$ contains a compact open subgroup, and we let $\tilde{U}$ be any algebraic complement of $\tilde{f}_0$ in $\tilde{G}$. By our assumption, $\langle x \rangle + G_0$ is a closed subgroup of G for $x \in \tilde{U}$, i.e. $\langle x + G_0 \rangle$ is closed in G/G_0 and hence discrete. Therefore, all compact elements of G/G_0 have finite order.

If the identity component of an LCA group G is compact, then it is evident that (i) is equivalent to the property that $\langle x\rangle$ is closed in G for every element x of any *fixed* algebraic complement of G_0 in G. However, this need not be true if G_0 is not compact: Let η be a monomorphism from J_p (p-adic integers) into $\boldsymbol{R}$ and $\bar{x}\in J_p$. $\langle(\eta(\bar{x}), \bar{x})\rangle \subseteq \{(\eta(x), x): x \in J_p\}$ is a closed subgroup of $\boldsymbol{R} \times J_p$, but J_p is compact and torsion-free.

The main result of this paper establishes a necessary and sufficient condition on an LCA group G in order that all elements of G/G_0 are compact and G_0 splits in G. We first need a preliminary lemma.

LEMMA 3. *Let N be a subgroup of a discrete group G. Then $(tG+N)/N$ is the torsion part of G/N if and only if $\hat{G}_0 \cap (\hat{G}, N)$ is the identity component of $(\hat{G}, N)$.*

PROOF. Assume $(\hat{G}, N)_0 = \hat{G}_0 \cap (\hat{G}, N)$, let $\varphi: G \to G/N$ be the natural map and ρ the topological isomorphism from $(G/N)^\frown$, onto $(\hat{G}, N)$ induced by φ. Clearly ρ maps the identity component of $(G/N)^\frown$, which is the annihilator of $t(G/N)$ in $(G/N)^\frown$ onto $(\hat{G}, N)_0$. Since ρ also establishes the isomorphism between $((G/N)^\frown, (tG+N)/N)$ and $(\hat{G}, tG + N) = (\hat{G}, tG) \cap (\hat{G}, N)$, we get $((G/N)^\frown, t(G/N)) = ((G/N)^\frown, (tG + N)/N)$. Thus $t(G/N)$ coincides with $(tG+N)/N$, as claimed. The converse is proved similary.

Robertson [7] calls an LCA group G a *topological torsion group*, if $(n!)x \to 0$ for each $x \in G$. We note the useful fact (see [7], 3.15) that G is a topological torsion group if and only if both G and $\hat{G}$ are totally disconnected.

THEOREM 4. *Let G be an* LCA *group. Then G is a topological torsion group modulo G_0 and contains G_0 as a direct summand if and only if there exists a descending chain $B^1 \supseteq \dots \supseteq B^n \supseteq \dots$ of compact subgroups of G such that*

(i) $\bigcap_{n\in N} B^n = \{0\}$;

(ii) $G/(G_0 + B^n)$ *is a torsion group for every* n;

(iii) $(B^n)_0 = G_0 \cap B^n$ *for every* n.

PROOF. First assume that G/G_0 is a topological torsion group. If G_0 splits in G, let C be any complement of G_0 in G. Hence, there is a compact open subgroup K of C and we define $B^n = n!\,K$ for every positive integer n. Since $\bigcap_{n\in N} n!\,K = K_0$, we get (i) and since the discrete group C/K is a torsion group, (ii) is also true. (iii) is obvious.

Conversely, let G contain compact subgroups $B^1 \supseteq \dots \supseteq B^n \supseteq \dots$ satisfying (i)-(iii). Suppose $x \in G$, where $\overline{\langle x + G_0 \rangle} \subseteq G/G_0$ is discrete. Since the intersection of $\langle x + G_0 \rangle$ and $(B^1 + G_0)/G_0$ is discrete and compact, it is finite, so we have by (ii) that $\langle x + G_0 \rangle$ is finite. Thus, all elements of G/G_0 are compact, i.e., G/G_0 is a topological torsion group. If G is compact, then we define $A_n = (\hat{G}, B^n)$ for every n. Since $\sum_{n \in N} A_n = (\hat{G}, \bigcap_{n \in N} B^n)$, the discrete group $\hat{G}$ is the union of the ascending chain $A_1 \subseteq \dots \subseteq A_n \subseteq \dots$ of subgroups. For all n, the compact group $G/(G_0 + B^n)$ is bounded and its dual group is isomorphic to tA_n; hence tA_n is bounded. Furthermore we obtain $t(\hat{G}/A_n) = (t\hat{G} + A_n)/A_n$ by Lemma 3. Therefore, $\hat{G}$ contains a splitting torsion part (cf. Fuchs [3], 100.4), and duality shows that G_0 splits in G. Now let G be arbitrary and write it as $V \oplus \tilde{G}$, where V is a maximal vector subgroup and $\tilde{G}$ has a compact open subgroup G'. Define $C^n = B^n \cap G'$ for every n. It is clear that the subgroups C^n have trivial intersection, and each group $G'/(G_0' + C^n)$ is torsion. Moreover, we get $(C^n)_0 = G_0' \cap C^n$ for all n. By what we have shown above, G_0' splits in G'. The passage from the compact open subgroup G' to the group G is as in the proof of Proposition 1.

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