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Kevin K. Blount
Sacred Heart University

Constantine Tsinakis
Vanderbilt University

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THE STRUCTURE OF RESIDUATED LATTICES

KEVIN BLOUNT

Department of Mathematics, Sacred Heart University
Fairfield, CT 06432

CONSTANTINE TSINAKIS

Department of Mathematics, Vanderbilt University
Nashville, TN 37240

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A **residuated lattice** is an ordered algebraic structure

$$\mathbf{L} = \langle L, \wedge, \vee, \cdot, \mathbf{e}, \backslash, / \rangle$$

such that $\langle L, \wedge, \vee \rangle$ is a lattice, $\langle L, \cdot, \mathbf{e} \rangle$ is a monoid, and $\backslash$ and $/$ are binary operations for which the equivalences

$$a \cdot b \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a \backslash c$$

hold for all $a, b, c \in L$. It is helpful to think of the last two operations as left and right division and thus the equivalences can be seen as "dividing" on the right by b and "dividing" on the left by a. The class of all residuated lattices is denoted by $\mathcal{RL}$.

The study of such objects originated in the context of the theory of ring ideals in the 1930s. The collection of all two-sided ideals of a ring forms a lattice upon which one can impose a natural monoid structure making this object into a residuated lattice. Such ideas were investigated by Morgan Ward and R. P. Dilworth in a series of important papers [15, 16, 45–48] and also by Krull in [33]. Since that time, there has been substantial research regarding some specific classes of residuated structures, see for example [1, 9, 26] and [38], but we believe that this is the first time that a general structural theory has been established for the class $\mathcal{RL}$ as a whole. In particular, we develop the notion of a normal subalgebra and show that $\mathcal{RL}$ is an "ideal variety" in the sense that it is an equational class in which congruences correspond to "normal" subalgebras in the same way that ring congruences correspond to ring ideals. As an application of the general theory, we produce an equational basis for the important subvariety $\mathcal{RL}^{\mathcal{C}}$ that is generated by all residuated chains. In the process, we find that this subclass has some remarkable structural properties that we believe could lead to some important decomposition theorems for its finite members (along the lines of the decompositions provided in [27]).

Keywords: Residuated lattice; residuated partially-ordered monoid; lattice-ordered group; relatively normal lattice.

AMS Mathematics Subject Classification: 06B05, 06B10, 06B20, 03B20, 03B50, 03B52, 03B70

1. Introduction

Our aim in this paper is to lay the groundwork for, and provide some significant initial contributions to, the development of a comprehensive theory on the structure of residuated lattices — a class of algebraic structures that we shall denote $\mathcal{RL}$. We believe that such a theory, whether in part or in whole, is not only fascinating in its own right, but also establishes a common framework within which researchers from a host of diverse disciplines can find tools and models applicable to their own areas.

The defining properties that describe the class $\mathcal{RL}$ are few and easy to quickly grasp. Moreover, one can readily construct concrete examples that illustrate the key features of such structures. However, the theory is also sufficiently robust that the class of residuated lattices encompasses a surprising number of topics from subjects as disparate as ℓ-groups, algebraic logic, and some areas of theoretical computer science. Even the objects constructed by Prenowitz [42] and others in their algebraic treatment of Euclidean geometry give rise to special types of residuated structures. We show in a few special instances that we are able to take guidance from some of these areas and generalize known results in their realm to the entire class $\mathcal{RL}$.

It is easy to see that the equivalences which define residuation can be captured by equations and thus $\mathcal{RL}$ is a finitely based variety. In order to emphasize the large number of important classes that are contained within $\mathcal{RL}$, we give in Fig. 1 a partial sketch of its lattice of subvarieties — henceforth denoted $\mathbf{L}(\mathcal{RL})$. The line segments in the diagram are intended to convey the relative positions of the indicated subclasses and we do not mean to imply that this fragment is a sublattice of $\mathbf{L}(\mathcal{RL})$.

Here, $\mathcal{RL}^{\mathcal{C}}$ denotes the subvariety of $\mathcal{RL}$ generated by all residuated chains, $\mathcal{B}r$ the variety of Brouwerian (or Heyting) algebras (in the sense of Köhler [32]), $\mathcal{R}SA$ the variety of relative Stone algebras and $\mathcal{B}A$ the variety of generalized Boolean

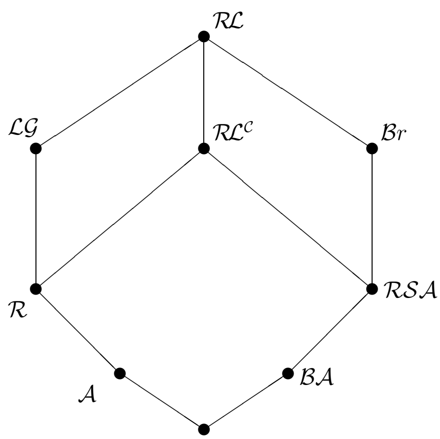

Fig. 1. A fragment of the subvariety lattice of $\mathcal{RL}$.

algebras (that is, relatively complemented, distributive lattices with a greatest element). Following the notational conventions of [1], $\mathcal{L}G$ is the variety of all lattice-ordered groups (ℓ-groups), $\mathcal{R}$ is the variety of representable ℓ-groups and $\mathcal{A}$ is the variety of all Abelian ℓ-groups. Of course, we are being lax with regard to the similarity types of these various varieties. Thus by $\mathcal{B}r$, for example, we mean the subvariety of $\mathcal{RL}$ generated by the additional equation $xy \approx x \wedge y$, and by $\mathcal{R}SA$ we mean the subvariety of $\mathcal{B}r$ generated by the equation $(x\backslash y) \vee (y\backslash x) = \mathbf{e}$. Similarly, the other classes are equationally defined subvarieties in the language of $\mathcal{RL}$; the point is that each of these is term-equivalent to its namesake variety and thus we feel justified in using the same cognomen without danger of confusion. In several of these subvarieties the right and left division operations correspond to already familiar notions. For example, the members of $\mathcal{B}r$ (which are the models of intuitionistic logic) all satisfy the equation $y/x = x\backslash y$ and this common value is usually denoted by $x \to y$, where $\to$ is the so-called Heyting arrow.

While previous research by others has thoroughly described in detail several particular classes of residuated structures, we present here a number of general results that hold throughout the variety $\mathcal{RL}$. We conclude our introduction with a brief outline of those results.

For $\mathbf{L} \in \mathcal{RL}$ and fixed $a \in L$ we define the notion of *right and left conjugation by* a: $\lambda_a(x) := [a\backslash(xa)] \wedge \mathbf{e}$ and $\rho_a(x) := [(ax)/a] \wedge \mathbf{e}$ respectively (the factor $\wedge\, \mathbf{e}$ appears for essentially technical reasons). These are unary operations on the universe of $\mathbf{L}$ that correspond to the analogous concepts from group theory. A subalgebra of $\mathbf{L}$ is called *normal* if it is closed with respect to all conjugations and it is said to be *convex* if it is order-convex with respect to the lattice ordering on $\mathbf{L}$. We let $\mathcal{C}N(\mathbf{L})$ denote the collection of all convex normal subalgebras of $\mathbf{L}$ and in Sec. 4 we establish that $\mathcal{RL}$ is an ideal variety:

Theorem 4.12 *For any* $\boldsymbol{L} \in \mathcal{RL}$, $\mathrm{Con}(\boldsymbol{L}) \cong \mathcal{C}N(\boldsymbol{L})$.

In Sec. 6, we give an explicit basis for the subvariety, $\mathcal{RL}^{\mathcal{C}}$, namely:

Theorem 6.7 $\mathcal{RL}^{\mathcal{C}} = \mathrm{Mod}_{\mathcal{RL}}[\varepsilon_1 \wedge \varepsilon_2]$ where ε_1 and ε_2 are the equations

$$\varepsilon_1 : (x \vee y) \wedge \mathbf{e} = (x \wedge \mathbf{e}) \vee (y \wedge \mathbf{e})\,.$$

$$\varepsilon_2 : \lambda_z[x/(x \vee y)] \vee \rho_w[y/(x \vee y)] = \mathbf{e}\,.$$

In the process of establishing these two theorems we provide element-wise descriptions of convex normal subalgebras and submonoids generated by arbitrary subsets. Finally, we investigate some further properties of the subvariety $\mathcal{RL}^{\mathcal{C}}$ — a class that we believe is particularly interesting for several reasons. For example, it follows from the work of Tsinakis and Hart [27] that for $\mathbf{L} \in \mathcal{RL}^{\mathcal{C}}$, the compact elements of $\mathrm{Con}(\mathbf{L})$ form a *relatively normal* lattice.

2. Preliminaries

We presume that the reader is familiar with the basic facts, definitions and terminology from universal algebra and lattice theory. In particular, the notions of **posets**, **lattices**, and **general algebras** are central to this paper as are the concepts of **congruences**, and **homomorphisms**. For an introduction to universal algebra and general algebraic systems, the reader may wish to consult [8] or [36] while any of [3, 5, 13, 23] or [24] would serve as a suitable lattice theory reference. Several of the results in this paper were motivated by analogous ideas in the theory of lattice-ordered groups and the reader interested in this topic may wish to see [1, 21] or [22].

If $\mathbf{P}$ is a poset, $X \subseteq P$ and $p \in P$ then we use the following notational conventions:

The **principal downset** of p in $\mathbf{P}$ is the set

$$\downarrow p := \{x \in P | x \leq p\}.$$

The **downset generated by** X in $\mathbf{P}$ is the set

$$\downarrow X := \{p \in P | p \leq x \text{ for some } x \in X\}.$$

A set X is called a **downset** or **order ideal** of $\mathbf{P}$ if $\downarrow X = X$.

The **dual** of a poset $\mathbf{P}$ is the poset $\mathbf{P}^{\partial}$ whose underlying set is the set P and whose ordering is just the opposite of that in $\mathbf{P}$. We also have the dual notions of those listed above, defined in the obvious ways:

The **principal upset** of p in $\mathbf{P}$, denoted by $\uparrow p$, is the set $\downarrow p$ of $\mathbf{P}^{\partial}$.
The **upset generated by** X in $\mathbf{P}$, denoted by $\uparrow X$, is the set $\downarrow X$ of $\mathbf{P}^{\partial}$.
A set X is called an **upset** or **order filter** of $\mathbf{P}$ if $\uparrow X = X$.

We shall denote the **bottom element** of a poset $\mathbf{P}$, if it exists, by $0_{\mathbf{P}}$. Similarly, $\top_{\mathbf{P}}$ denotes the **top element**. Obviously, bottom elements and top elements, when they exist, are unique. Let $X \subseteq P$ be any subset (possibly empty). We will use $\bigvee_{\mathbf{P}} X$ and $\bigwedge_{\mathbf{P}} X$, respectively, to denote the **supremum** (or least upper bound) and **infimum** (or greatest lower bound) of X in $\mathbf{P}$ whenever they exist. We will use the terms **monotone**, **isotone**, and **order-preserving** synonymously to describe a map $f\colon P \to Q$ between posets $\mathbf{P}$ and $\mathbf{Q}$ with the property that for all $p_1, p_2 \in P$, if $p_1 \leq p_2$ then $f(p_1) \leq f(p_2)$. If for all $p_1, p_2 \in P$, $p_1 \leq p_2 \Rightarrow f(p_1) \geq f(p_2)$, then f will be called **anti-isotone** or **order-reversing**. The poset subscripts appearing in some of the notation of this paragraph will henceforth be omitted whenever there is no danger of confusion.

3. Residuated Lattices

Let P be a poset and $\cdot\colon P \times P \to P$ be a binary map. We say that $\cdot$ is *residuated* provided there exist binary maps $\backslash\colon P \times P \to P$ and $/\colon P \times P \to P$ such that

$$x \cdot y \le z \Leftrightarrow x \le z/y \Leftrightarrow y \le x\backslash z\,,$$

for all $x, y, z \in P$. The maps $\backslash$ and $/$ are called the *residuals* of $\cdot$. Note that a binary operation is residuated if and only if it is order preserving in both variables and for all $a, b \in P$, the sets $\{p \in P | ap \le b\}$ and $\{p \in P | pa \le b\}$ both contain largest elements. As a consequence of the general theory of adjunctions (see [19]), multiplication preserves all existing joins in each argument. Moreover, the residual operations preserve all existing meets in the "numerator" and convert all existing joins in the "denominator" to meets. See Lemma 3.2 below.

Definition 3.1. A **residuated lattice-ordered monoid**, or a **residuated lattice** for short, is an algebraic system

$$\mathbf{L} = \langle L, \wedge, \vee, \cdot, \mathbf{e}, \backslash, / \rangle$$

such that $\langle L, \wedge, \vee \rangle$ is a lattice; $\langle L, \cdot, \mathbf{e} \rangle$ is a monoid; and $\langle \backslash, / \rangle$ are the residuals of $\cdot$ in the lattice order.

We will use the symbol $\mathcal{RL}$ to denote the class of all residuated lattices. Note that some authors omit the constant $\mathbf{e}$ from the definition and refer to those residuated lattices with unit as **unital**. Also, we adopt the usual convention of representing the monoid operation by juxtaposition, writing ab for $a \cdot b$.

The following lemma collects numerous basic properties of residuated lattices, most of which by now can be ascribed to the subject's "folklore". Notice that items 2 and 3 imply that the division operations are isotone in the numerator and anti-isotone in the denominator. We leave the proofs to the reader since they are routine.

Lemma 3.2. *Let $\mathbf{L}$ be a residuated lattice. For all $a, b, c \in L$, and any $Y \subseteq L$, we have*:

1. (a) $a(b \vee c) = ab \vee ac$ *and*
   $(b \vee c)a = ba \vee ca$.
   (b) *If* $\bigvee Y$ *exists, then*
   $$a\Big(\bigvee Y\Big) = \bigvee\{ay | y \in Y\} \text{ and}$$
   $$\Big(\bigvee Y\Big)a = \bigvee\{ya | y \in Y\}\,.$$
2. (a) $(a \wedge b)/c = (a/c) \wedge (b/c)$ *and* $c\backslash(a \wedge b) = (c\backslash a) \wedge (c\backslash b)$.
   (b) *If* $\bigwedge Y$ *exists, then*
   $$\Big(\bigwedge Y\Big)\Big/c = \bigwedge\{y/c | y \in Y\} \text{ and}$$
   $$c\Big\backslash\Big(\bigwedge Y\Big) = \bigwedge\{c\backslash y | y \in Y\}\,.$$

3. (a) $a/(b \vee c) = (a/b) \wedge (a/c)$ *and* $(b \vee c)\backslash a = (b\backslash a) \wedge (c\backslash a)$.
   (b) *If* $\bigvee Y$ *exists, then*

$$a\Big/\Big(\bigvee Y\Big) = \bigwedge\{a/y|y \in Y\} \text{ and}$$

$$\Big(\bigvee Y\Big)\Big\backslash a = \bigwedge\{y\backslash a|y \in Y\}\,.$$

4. $(a/c)c \leq a$ *and* $c(c\backslash a) \leq a$.
5. $a(c/b) \leq ac/b$ *and* $(a\backslash c)b \leq a\backslash cb$.
6. $(c/b)(b/a) \leq c/a$ *and* $(a\backslash b)(b\backslash c) \leq a\backslash c$.
7. $c/b \leq (c/a)/(b/a)$ *and* $b\backslash c \leq (a\backslash b)\backslash(a\backslash c)$.
8. $b/a \leq (c/b)\backslash(c/a)$ *and* $a\backslash b \leq (a\backslash c)/(b\backslash c)$.
9. $c/b \leq ca/ba$ *and* $a\backslash c \leq ba\backslash bc$
10. $(c/a)/b = c/ba$ *and* $b\backslash(a\backslash c) = ab\backslash c$.
11. $a\backslash(c/b) = (a\backslash c)/b$.
12. $c \leq (a/c)\backslash a$ *and* $c \leq a/(c\backslash a)$.
13. $a/\mathbf{e} = a$ *and* $\mathbf{e}\backslash a = a$.
14. $a/a \geq \mathbf{e}$ *and* $a\backslash a \geq \mathbf{e}$.
15. $(a/b)(\mathbf{e}/c) \leq a/cb$ *and* $(c\backslash \mathbf{e})(b\backslash a) \leq bc\backslash a$.
16. $(a/a)a = a$ *and* $a(a\backslash a) = a$.
17. $(a/a)^2 = a/a$ *and* $(a\backslash a)^2 = a\backslash a$.
18. *If* $\boldsymbol{L}$ *has a bottom element*, 0, *then* $\boldsymbol{L}$ *also has a top element*, $\top$, *and for all* $a \in L$ *we have*:
    (a) $a0 = 0a = 0$.
    (b) $a/0 = 0\backslash a = \top$.
    (c) $\top/a = a\backslash\top = \top$.

4. The Class $\mathcal{RL}$ is an Ideal Variety

By an **ideal variety** we mean an equational class of algebras with the property that for each member $\mathbf{A}$, the congruences of $\mathbf{A}$ correspond to certain subalgebras of $\mathbf{A}$. The meaning of this term will be clarified throughout the remainder of the paper; for a precise definition see [25] or [44]. We begin by showing the well-known fact that $\mathcal{RL}$ is indeed an equational class.

Proposition 4.1. *The class* $\mathcal{RL}$ *is a finitely based equational class. In particular*, $\mathcal{RL} = \mathrm{Mod}(\Sigma)$ *where* Σ *consists of the defining equations for lattices and monoids together with the six equations given below*:

$$\begin{array}{ll} a \leq (ab \vee c)/b & b \leq a\backslash(ab \vee c) \\ a(b \vee c) = ab \vee ac & (b \vee c)a = ba \vee ca \\ (a/b)b \leq a & b(b\backslash a) \leq a \end{array}$$

Proof. Suppose $\mathbf{L} \in \mathcal{RL}$. Then for any $a, b, c \in L$,

$$(ab \vee c)/b \geq ab/b \geq a(b/b) \geq a$$

so that **L** satisfies the first equation above. That **L** satisfies both $a(b \vee c) = ab \vee ac$ and $(a/b)b \leq a$ follows from Lemma 3.2. The three dual equations are proved to hold in a similar manner. Now suppose **L** is an algebra in the language of residuated lattices and $\mathbf{L} \models \Sigma$. Then we have that **L** is a lattice with respect to the meet and join symbols and a monoid under the multiplication symbol with unit equal to the constant symbol. It only remains to prove that **L** satisfies the equivalences

$$ab \leq c \Leftrightarrow b \leq a\backslash c \Leftrightarrow a \leq c/b\,.$$

Suppose then that $ab \leq c$. From $a \leq (ab \vee c)/b$ we deduce that $a \leq c/b$. Conversely, suppose that $a \leq c/b$. From $a(b \vee c) = ab \vee ac$ we see that multiplication preserves order so that $ab \leq (c/b)b$. Finally $(a/b)b \leq a$ gives us the desired conclusion. The other equivalence is proved similarly. □

Definition 4.2. If **L** is a residuated lattice, the set $L^{-} := \{a \in L | a \leq \mathbf{e}\}$ is called the **negative cone** of **L**.

Note that the negative cone is a submonoid of $\langle L, \cdot, \mathbf{e}\rangle$. As such, we will denote it by $\mathbf{L}^{-}$.

Definition 4.3. Let $\mathbf{L} \in \mathcal{RL}$. For each $a \in L$, define $\rho_a(x) = (ax/a) \wedge \mathbf{e}$ and $\lambda_a(x) = (a\backslash xa) \wedge \mathbf{e}$. We refer to ρ_a and λ_a respectively as **right** and **left conjugation** by a.

Let $\mathrm{P} = \{\rho_a | a \in L\}$, $\Lambda = \{\lambda_a | a \in L\}$ and set

$$\Gamma = \{\gamma | \exists n, \text{ and } \exists \gamma_j \in (\mathrm{P} \cup \Lambda) \text{ so that } \gamma = \gamma_1 \circ \gamma_2 \circ \ldots \circ \gamma_n\}\,.$$

We will call each $\gamma \in \Gamma$ an **iterated conjugation** map.

Definition 4.4. A subset $X \subseteq L$ is called **convex** if for any $x, y \in X$ and $a \in L$, $x \leq a \leq y \Rightarrow a \in X$; X is called **normal** if X is closed with respect to all $\rho \in \mathrm{P}$ and $\lambda \in \Lambda$.

Note that a subset is normal if and only if it is closed with respect to all $\gamma \in \Gamma$.

Definition 4.5. Let **L** be a residuated lattice. For $a, b \in L$ define $[a, b]_r = (ab/ba) \wedge \mathbf{e}$ and $[a, b]_l = (ba\backslash ab) \wedge \mathbf{e}$. We call $[a, b]_r$ and $[a, b]_l$ respectively the **right** and **left commutators** of a with b.

We will say that a subset X is *closed with respect to commutators* if for any $a \in L$ and $x \in X$, the commutators $[a, x]_r$ and $[x, a]_l$ both lie in X. Normality and "closure with respect to commutators" are identical properties for certain "nice" subsets as we show in the next two lemmas.

Lemma 4.6. *Let **H** be a convex subalgebra of **L**. Then **H** is normal if and only if **H** is closed with respect to commutators.*

Proof. Suppose $\mathbf{H}$ is normal. Then

$$\mathbf{e} \geq [a,h]_r = (ah/ha) \wedge \mathbf{e} = ((ah/a)/h) \wedge \mathbf{e}$$
$$\geq (((ah/a) \wedge \mathbf{e})/h) \wedge \mathbf{e} = (\rho_a(h)/h) \wedge \mathbf{e} \in H$$

so that $[a,h]_r \in H$ by convexity. The proof that $[h,a]_l \in H$ is analogous.

Conversely, suppose H is closed with respect to commutators. Then

$$[a,h]_r h \wedge \mathbf{e} \in H \text{ and } [a,h]_r h \wedge \mathbf{e}$$
$$= ((ah/ha) \wedge \mathbf{e})h \wedge \mathbf{e} \leq (ah/ha)h \wedge \mathbf{e}$$
$$= ((ah/a)/h)h \wedge \mathbf{e} \leq (ah/a) \wedge \mathbf{e} = \rho_a(h) \leq \mathbf{e}$$

so $\rho_a(h) \in H$ by convexity. The proof that $\lambda_a(h) \in H$ is analogous. □

The same result holds for convex submonoids of the negative cone of $\mathbf{L}$:

Lemma 4.7. *If $\boldsymbol{S}$ is a convex submonoid of $\boldsymbol{L}^-$, then $\boldsymbol{S}$ is normal if and only if S is closed with respect to commutators.*

Proof. Let $s \in S$ and $a \in L$ and suppose $\mathbf{S}$ is normal. Then

$$\mathbf{e} \geq [a,s]_r = (as/sa) \wedge \mathbf{e} = ((as/a)/s) \wedge \mathbf{e} \geq (as/a) \wedge \mathbf{e} = \rho_a(s) \in S$$

where the last inequality above follows since $s \leq \mathbf{e}$. Similarly, $[s,a]_l \in S$. Conversely, if S is closed with respect to commutators, then $[a,s]_r s \in S$. But

$$[a,s]_r s = (((as/a)/s) \wedge \mathbf{e})s \leq ((as/a)/s)s \wedge s$$
$$\leq (as/a) \wedge s \leq (as/a) \wedge \mathbf{e} = \rho_a(s) \leq \mathbf{e}$$

and by convexity we have $\rho_a(s) \in S$. Similarly, $\lambda_a(s) \in S$. □

4.1. *Two "switching" identities*

We often find it useful to convert one of the division operations into its dual. The following two identities, which can be verified by straightforward calculation, provide a means by which to do so in any residuated lattice:

$$z/y \leq py\backslash z\,, \quad \text{where } p = [z/y, y]_r\,, \quad \text{and}$$
$$x\backslash z \leq z/xq\,, \quad \text{where } q = [x, x\backslash z]_l\,.$$

Note: the above identities still hold if the "$\wedge\mathbf{e}$" factor is omitted from the commutators.

Lemma 4.8. *Let $\boldsymbol{L}$ be a residuated lattice and $\theta \in \mathrm{Con}(\boldsymbol{L})$. Then the following are equivalent*:

1. $a\theta b$

2. $[(a/b) \wedge \mathbf{e}]\theta\mathbf{e}$ *and* $[(b/a) \wedge \mathbf{e}]\theta\mathbf{e}$
3. $[(a\backslash b) \wedge \mathbf{e}]\theta\mathbf{e}$ *and* $[(b\backslash a) \wedge \mathbf{e}]\theta\mathbf{e}$

Proof. Suppose $a\theta b$. Then $(a/a)\theta(b/a)$ so that

$$\mathbf{e} = [(a/a) \wedge \mathbf{e}]\theta[(b/a) \wedge \mathbf{e}]$$

and the other relations in items 2 and 3 follow similarly. Conversely, suppose both $[(a/b) \wedge \mathbf{e}]\theta\mathbf{e}$ and $[(b/a) \wedge \mathbf{e}]\theta\mathbf{e}$. Set $r = [(a/b) \wedge \mathbf{e}]b$ and $s = [(b/a) \wedge \mathbf{e}]a$. Then $r\theta b$ and $s\theta a$. Moreover, $r \leq (a/b)b \leq a$ and $s \leq (b/a)a \leq b$ so that $r = (a \wedge r)\theta(a \wedge b)$ and $s = (b \wedge s)\theta(b \wedge a)$ whence $b\theta r\theta(a \wedge b)\theta s\theta a$; we have shown item 2 $\Rightarrow$ item 1. One proves item 3 $\Rightarrow$ item 1 in an analogous manner. $\square$

Lemma 4.9. *Let θ be a congruence relation on a residuated lattice $\boldsymbol{L}$. Then $[\mathbf{e}]_\theta := \{a \in A | a\theta\mathbf{e}\}$ is a convex normal subalgebra of $\boldsymbol{L}$.*

Proof. Since $\mathbf{e}$ is idempotent with respect to all the binary operations of $\mathbf{L}$, it immediately follows that $[\mathbf{e}]_\theta$ forms a subalgebra of $\mathbf{L}$. Convexity is a consequence of the well-known fact that any block of any lattice congruence is convex. Finally, let $a \in [\mathbf{e}]_\theta$ and $c \in L$. Then

$$\lambda_c(a) = [c\backslash ac] \wedge \mathbf{e}\theta[c\backslash \mathbf{e}c] \wedge \mathbf{e} = [c\backslash c] \wedge \mathbf{e} = \mathbf{e}$$

so that $\lambda_c(a) \in [\mathbf{e}]_\theta$. Similarly, $\rho_c(a) \in [\mathbf{e}]_\theta$. $\square$

Lemma 4.10. *Suppose $\boldsymbol{H}$ is a convex normal subalgebra of $\boldsymbol{L}$. For any $a, b \in L$,*

$$(a/b) \wedge \mathbf{e} \in H \Leftrightarrow (b\backslash a) \wedge \mathbf{e} \in H\,.$$

Proof. Suppose $(a/b) \wedge \mathbf{e} \in H$. Since $\mathbf{H}$ is normal, we have

$$h := b\backslash([(a/b) \wedge \mathbf{e}]b) \wedge \mathbf{e} \in H\,.$$

But $h \leq [b\backslash(a/b)b] \wedge \mathbf{e} \leq (b\backslash a) \wedge \mathbf{e} \leq \mathbf{e} \in H$ so that $(b\backslash a) \wedge \mathbf{e} \in H$. The reverse implication is proved similarly. $\square$

Next we characterize the congruence corresponding to a given convex normal subalgebra (see [35] in which McCarthy gives a similar description for a related congruence in a special case).

Lemma 4.11. *Let $\boldsymbol{H}$ be a convex normal subalgebra of a residuated lattice $\boldsymbol{L}$. Then*

$$\begin{aligned}\theta_{\boldsymbol{H}} &:= \{(a,b) | \exists h \in H, ha \leq b \text{ and } hb \leq a\} \\ &= \{(a,b) | (a/b) \wedge \mathbf{e} \in H \text{ and } (b/a) \wedge \mathbf{e} \in H\} \\ &= \{(a,b) \mid (a\backslash b) \wedge \mathbf{e} \in H \text{ and } (b\backslash a) \wedge \mathbf{e} \in H\}\end{aligned}$$

is a congruence on $\boldsymbol{L}$.

Proof. First we show that the three sets defined above are indeed equal. That the second and third sets are identical follows from Lemma 4.10. If (a,b) is a member of the second set, then letting $h = (a/b) \wedge (b/a) \wedge \mathbf{e}$ we have $h \in H$ and

$$ha \leq (b/a)a \leq b \text{ and } hb \leq (a/b)b \leq a$$

so that (a,b) is a member of the first set. Conversely, if (a,b) is a member of the first set then for some $h \in H$ we have

$$ha \leq b \Rightarrow h \leq b/a \Rightarrow h \wedge \mathbf{e} \leq (b/a) \wedge \mathbf{e} \leq \mathbf{e}$$

and by convexity, we conclude that $(b/a) \wedge \mathbf{e} \in H$. Similarly, $(a/b) \wedge \mathbf{e} \in H$.

We now prove $\theta_{\mathbf{H}}$ is a congruence using the second set as our description.

$\boldsymbol{\theta}$ is an equivalence relation: Note that θ is reflexive since for any $a \in L$ we have $(a/a) \wedge \mathbf{e} = \mathbf{e} \in H$ and θ is symmetric by the symmetry of its definition. Finally, to see that θ is transitive, suppose $a\theta b$ and $b\theta c$. Then,

$$[(a/b) \wedge \mathbf{e}][(b/c) \wedge \mathbf{e}] \leq [(a/b)(b/c)] \wedge \mathbf{e} \leq (a/c) \wedge \mathbf{e} \leq \mathbf{e}$$

so that $(a/c) \wedge \mathbf{e} \in H$ since $\mathbf{H}$ is convex. Similarly, $(c/a) \wedge \mathbf{e} \in H$ so $a\theta c$.

$\boldsymbol{\theta}$ is compatible with multiplication: Suppose $a\theta b$ and $c \in L$. Then

$$(a/b) \wedge \mathbf{e} \leq (ac/bc) \wedge \mathbf{e} \leq \mathbf{e}$$

so $(ac/bc) \wedge \mathbf{e} \in H$. Similarly, $(bc/ac) \wedge \mathbf{e} \in H$ so that $(ac)\theta(bc)$. Next, using the normality of $\mathbf{H}$,

$$\rho_c((a/b) \wedge \mathbf{e}) = (c[(a/b) \wedge \mathbf{e}]/c) \wedge \mathbf{e} \in H\,.$$

But

$$\rho_c((a/b) \wedge \mathbf{e}) \leq [c(a/b)/c] \wedge \mathbf{e} \leq [ca/b/c] \wedge \mathbf{e} = (ca/cb) \wedge \mathbf{e} \leq \mathbf{e} \in H$$

so that $(ca/cb) \wedge \mathbf{e} \in H$. Similarly, $(cb/ca) \wedge \mathbf{e} \in H$ so that $(ca)\theta(cb)$.

$\boldsymbol{\theta}$ is compatible with meet: Suppose $a\theta b$ and $c \in L$. Set $r = (a/b) \wedge \mathbf{e}$. Since $r \leq 1$ we have $rc \leq c$; also $r \leq a/b$ gives $rb \leq a$. Thus,

$$r(b \wedge c) \leq (rb) \wedge (rc) \leq a \wedge c\,.$$

From this it follows that $r \leq (a \wedge c)/(b \wedge c)$ which implies

$$r = (r \wedge \mathbf{e}) \leq [(a \wedge c)/(b \wedge c)] \wedge \mathbf{e} \leq \mathbf{e}$$

whence $[(a \wedge c)/(b \wedge c)] \wedge \mathbf{e} \in H$. Similarly, $[(b \wedge c)/(a \wedge c)] \wedge \mathbf{e} \in H$ so that $(a \wedge c)\theta(b \wedge c)$.

$\boldsymbol{\theta}$ is compatible with join: This proof is similar to (even easier than) the one above.

$\boldsymbol{\theta}$ is compatible with right division: Suppose $a\theta b$ and $c \in L$. Then we have

$$(a/b) \wedge \mathbf{e} \leq [(a/c)/(b/c)] \wedge \mathbf{e} \leq \mathbf{e}$$

so that $[(a/c)/(b/c)] \wedge \mathbf{e} \in H$. Similarly, $[(b/c)/(a/c)] \wedge \mathbf{e} \in H$ so that $(a/c)\theta(b/c)$. Next,

$$(b/a) \wedge \mathbf{e} \leq [(c/b)\backslash(c/a)] \wedge \mathbf{e} \leq \mathbf{e} \in H$$

so that $[(c/b)\backslash(c/a)] \wedge \mathbf{e} \in H$ and, by Lemma 4.10, $[(c/b)/(c/a)] \wedge \mathbf{e} \in H$. Similarly, $[(c/a)/(c/b)] \wedge \mathbf{e} \in H$ so that $(c/a)\theta(c/b)$.

$\boldsymbol{\theta}$ is compatible with left division: This proof is analogous to the one above. □

Theorem 4.12. *The lattice $\mathcal{C}N(\boldsymbol{L})$ of convex normal subalgebras of a residuated lattice $\boldsymbol{L}$ is isomorphic to its congruence lattice* $\mathrm{Con}(\boldsymbol{L})$. *The isomorphism is given by the mutually inverse maps $\boldsymbol{H} \mapsto \theta_{\boldsymbol{H}}$ and $\theta \mapsto [\mathbf{e}]_\theta$.*

Proof. We have shown both that $\theta_{\mathbf{H}}$ is a congruence and that $[\mathbf{e}]_\theta$ is a member of $\mathcal{C}N(\mathbf{L})$, and it is clear that the maps $\mathbf{H} \mapsto \theta_{\mathbf{H}}$ and $\theta \mapsto [\mathbf{e}]_\theta$ are monotone. It remains only to show that these two maps are mutually inverse, since it will then follow that they are lattice homorphisms.

Given $\theta \in \mathrm{Con}(\mathbf{L})$, set $H = [\mathbf{e}]_\theta$; we must show that $\theta = \theta_{\mathbf{H}}$. But this is easy; using Lemma 4.8,

$$a\theta b \Leftrightarrow [((a/b) \wedge \mathbf{e})\theta\mathbf{e}$$

$$\text{and } ((b/a) \wedge \mathbf{e})\theta 1] \Leftrightarrow [((a/b) \wedge \mathbf{e}) \in H$$

$$\text{and } ((b/a) \wedge \mathbf{e}) \in H] \Leftrightarrow a\theta_{\mathbf{H}} b\,.$$

Conversely, for any $\mathbf{H} \in \mathcal{C}N(\mathbf{L})$ we must show that $H = [\mathbf{e}]_{\theta_{\mathbf{H}}}$. But

$$h \in H \Rightarrow [(h/\mathbf{e}) \wedge \mathbf{e} \in H \text{ and } (\mathbf{e}/h) \wedge \mathbf{e} \in H]$$

so that $h \in [\mathbf{e}]_{\theta_{\mathbf{H}}}$. If $a \in [\mathbf{e}]_{\theta_{\mathbf{H}}}$ then $(a, \mathbf{e}) \in \theta_{\mathbf{H}}$ and we use the first description of $\theta_{\mathbf{H}}$ in Lemma 4.11 to conclude there exist some $h \in H$ such that $ha \leq \mathbf{e}$ and $h = h \cdot \mathbf{e} \leq a$. Now it follows from the convexity of $\mathbf{H}$ that $h \leq a \leq h\backslash\mathbf{e} \Rightarrow a \in H$. □

5. Subalgebra Generation

In the previous section we saw that the congruences of a residuated lattice $\mathbf{L}$ correspond to its convex normal subalgebras. Here we show that these subalgebras in turn correspond to the convex normal submonoids of $\mathbf{L}^-$. Thus, letting $\mathcal{C}N(\mathbf{L})$ and $\mathcal{C}NM(\mathbf{L}^-)$ denote respectively the lattices of convex normal subalgebras of $\mathbf{L}$ and convex normal submonoids of $\mathbf{L}^-$, we conclude that $\mathrm{Con}(\mathbf{L}) \cong \mathcal{C}N(\mathbf{L}) \cong \mathcal{C}NM(\mathbf{L}^-)$. Finally, we describe the convex normal subalgebra generated by an arbitrary subset $S \subseteq L$.

Our next theorem shows that a convex normal subalgebra is completely determined by its negative cone:

Theorem 5.1. *Let $\boldsymbol{S}$ be a convex normal submonoid of $\boldsymbol{L}^{-}$. Then defining the set H_S by*

$$H_S := \{a | s \leq a \leq s\backslash\mathbf{e} \textit{ for some } s \in S\}\,,$$

$\boldsymbol{H}_S$ is a convex normal subalgebra of $\boldsymbol{L}$ and $S = H_S^{-}$. Conversely, if $\boldsymbol{H}$ is any convex normal subalgebra of $\boldsymbol{L}$ the, setting $S_H = H^{-}$, $\boldsymbol{S}_H$ is a convex normal submonoid of $\boldsymbol{L}^{-}$ and H can be recovered from S_H as described above. Moreover, the mutually inverse maps $\boldsymbol{H} \mapsto \boldsymbol{S}_H$ and $\boldsymbol{S} \mapsto \boldsymbol{H}_S$ establish a lattice isomorphism between $\mathcal{CN}(\boldsymbol{L})$ and $\mathcal{CNM}(\boldsymbol{L}^{-})$.

Proof. Given a convex, normal subalgebra $\mathbf{H}$ of $\mathbf{L}$, the assertions about S_H are easy to verify. Thus we turn our attention to the other direction: let $\mathbf{S}$ be a convex normal submonoid of $\mathbf{L}^{-}$ and define H_S as above. It is easy to show that H_S is convex and normal. Moreover, it is immediate that $H_S^{-} = S$. However, we must verify that H_S is a subuniverse. Clearly $\mathbf{e} \in H_S$, so we check for closure under the binary operations: let $a, b \in H_S$. Then there are $s, t \in S$ so that

$$s \leq a \leq s\backslash\mathbf{e}\,, \quad \text{and}$$

$$t \leq b \leq t\backslash\mathbf{e}\,.$$

Closure under multiplication: Set $r = (st)(ts) \in S$. Then, by Lemma 3.2 item 15, we have

$$r \leq st \leq ab \leq (s\backslash\mathbf{e})(t\backslash\mathbf{e}) \leq ts\backslash\mathbf{e} \leq r\backslash\mathbf{e}\,.$$

Closure under meet: Set $r = st$. Then

$$r = st \leq s \wedge t \leq a \wedge b \leq (s\backslash\mathbf{e}) \wedge (t\backslash\mathbf{e}) \leq (r\backslash\mathbf{e}) \wedge (r\backslash\mathbf{e}) = r\backslash\mathbf{e}\,.$$

Closure under join: Similar to the above proof.
Closure under left division: We have

$$a\backslash b \leq s\backslash(t\backslash\mathbf{e}) = (ts)\backslash\mathbf{e}\,,$$

but to find a lower bound for $a\backslash b$ is a little trickier.

First notice that

$$t \leq b \text{ and } sa \leq \mathbf{e} \Rightarrow \; tsa \leq b\,.$$

From this we derive

$$ats(ats\backslash tsa) \leq tsa \leq b \Rightarrow ts(ats\backslash tsa) \leq a\backslash b\,.$$

Setting $p = (ats)\backslash(tsa)$ and $q = ts(p \wedge \mathbf{e})$, we know that $p \wedge \mathbf{e} = [ts, a]_l \in S$ and so $q \in S$. But now $q \leq tsp \leq a\backslash b$ and we have found the desired lower bound. Finally, setting $r = qts$, it follows that $r \leq a\backslash b \leq r\backslash\mathbf{e}$.

Closure under right division: First observe that

$$s \leq a \text{ and } tb \leq \mathbf{e} \Rightarrow \ stb \leq a \Rightarrow st \leq a/b\,,$$

but to find an upper bound is a little trickier:

$$a/b \leq (s\backslash\mathbf{e})/t \leq pt\backslash(s\backslash\mathbf{e}) = spt\backslash\mathbf{e}\,,$$

where

$$p = [(s\backslash\mathbf{e})/t, t]_r$$

as given by the switching identity. But $p \in S$ by the comments following Lemma 4.6 and we have found an appropriate upper bound. Finally, we can set $r = (st)(spt)$ and it follows that $r \leq a/b \leq r\backslash\mathbf{e}$.

We have shown that the maps between the two lattices are well-defined and mutually inverse. Since they are clearly isotone, the theorem is proved. □

The next two lemmas provide a description of the convex normal submonoid generated by an arbitrary subset of the negative cone.

Lemma 5.2. *For all $a_1, a_2, \ldots, a_n, b \in L$, if $a = \prod a_j$ then*

$$\prod \rho_b(a_j) \leq \rho_b(a)\,, \quad \textit{and} \prod \lambda_b(a_j) \leq \lambda_b(a)\,.$$

Proof. We prove only the case $n = 2$; the proof can be completed by the obvious induction.

$$\begin{aligned}\rho_b(a_1)\rho_b(a_2) &= [(ba_1/b) \wedge \mathbf{e}][(ba_2/b) \wedge \mathbf{e}] \leq [(ba_1/b)(ba_2/b)] \wedge \mathbf{e}\\ &\leq [((ba_1/b)ba_2)/b] \wedge \mathbf{e} \leq (ba_1a_2/b) \wedge \mathbf{e} = \rho_b(a_1a_2)\,.\end{aligned}$$

In the last two inequalities, we used Lemma 3.2 items 5 and 4 respectively. The proof for λ_b is analogous. □

Lemma 5.3. *Suppose $S \subseteq L^-$. Then the convex normal submonoid generated by S is $M(S)$, where $M(S)$ is constructed as follows.*
First, set

$$\hat{S} = \{\gamma(s) | s \in S, \gamma \in \Gamma\} \bigcup \{\mathbf{e}\}$$

and let

$$P(\hat{S}) = \textit{ all finite products of members of } \hat{S}\,.$$

Finally, define

$$M(S) = \{x | a \leq x \leq \mathbf{e} \textit{ for some } a \in P(\hat{S})\}\,.$$

Proof. It is clear that $\mathbf{e} \in M(S)$, that $M(S)$ is convex and closed under multiplication, and that any convex normal submonoid containing S must contain $M(S)$.

Moreover, since $S \subseteq L^{-}$, $S \subseteq M(S)$. It only remains to show that $M(S)$ is normal. But this follows from Lemma 5.2 and the convexity of $M(S)$: if $x \in M(S)$, then for some $a_1, a_2, \ldots, a_n \in \hat{S}$ and $a = \prod a_j$ we have $a \leq x \leq \mathbf{e}$. Moreover, for each j we have some $\gamma_j \in \Gamma$ and $s_j \in S$ so that $a_j = \gamma_j(s_j)$. For any $b \in L$, set $\gamma'_j = \rho_b \circ \gamma_j$, $a'_j = \gamma'_j(s_j)$ and $a' = \prod a'_j$. Then for each j, $a'_j \in \hat{S}$ whence $a' \in P(\hat{S})$. Finally, from Lemma 5.2, we have

$$a' = \prod a'_j = \prod \rho_b(a_j) \leq \rho_b(a) \leq \rho_b(x) \leq \rho_b(\mathbf{e}) = \mathbf{e}$$

and by the convexity of $M(S)$ we conclude that $\rho_b(x) \in M(S)$. An analogous proof gives $\lambda_b(x) \in M(S)$. $\square$

For any subset $S \subseteq L$, let $\mathcal{N}(S)$ denote the convex normal subalgebra generated by S.

Proposition 5.4. *If $S \subseteq L^{-}$, then*

$$\mathcal{N}(S) = \{x | a \leq x \leq a \backslash \mathbf{e} \textit{ for some } a \in P(\hat{S})\}\,.$$

Proof. This follows from the previous lemmas. Clearly

$$\mathcal{N}(S) = \{x | b \leq x \leq b \backslash \mathbf{e} \text{ for some } b \in M(S)\}\,.$$

But if $b \in M(S)$ then there is some $a \in P(\hat{S})$ so that $a \leq b \leq \mathbf{e}$ from which it follows that $a \leq x \leq a \backslash \mathbf{e}$. $\square$

A **principal** (convex normal) subalgebra is a (convex normal) subalgebra generated by a singleton; we will write $P(\hat{a})$ for $P(\hat{\{a\}})$ and $\mathcal{N}(a)$ for $\mathcal{N}(\{a\})$.

Lemma 5.5. *For any $a \in L$, $\mathcal{N}(a) = \mathcal{N}(a')$ where $a' = a \wedge (\mathbf{e}/a) \wedge \mathbf{e}$.*

Proof. Clearly $a' \in \mathcal{N}(a)$. On the other hand,

$$a' \leq a \leq (\mathbf{e}/a) \backslash \mathbf{e} \leq a' \backslash \mathbf{e}\,,$$

so that $a \in \mathcal{N}(a')$. $\square$

Thus we have the following corollaries:

Corollary 5.6. *If $a \in L$, then*

$$\mathcal{N}(a) = \{x | b \leq x \leq b \backslash \mathbf{e} \textit{ for some } b \in P(\hat{a'})\}\,,$$

where $a' = a \wedge (\mathbf{e}/a) \wedge \mathbf{e}$.

Corollary 5.7. *Let $S \subseteq L$ and set $S^* = \{s \wedge (\mathbf{e}/s) \wedge \mathbf{e} | s \in S\}$. Then*

$$\mathcal{N}(S) = \{x | a \leq x \leq (a \backslash \mathbf{e}), \textit{ for some } a \in P(\hat{S^*})\}\,.$$

6. The Subvariety $\mathcal{RL}^{\mathcal{C}}$

In this section we turn our attention to the subvariety of $\mathcal{RL}$ generated by all those residuated lattices that are totally ordered. Throughout this section, $\mathcal{C}$ will denote the class of all residuated chains and $\mathcal{K} \subseteq \mathcal{C}$ will be the class of all subdirectly irreducible (SI) members of $\mathcal{C}$. If Σ is a set of equations (or a single equation) in the language of $\mathcal{RL}$ then we will write $\mathrm{Mod}_{\mathcal{RL}}(\Sigma)$ to denote $\mathrm{Mod}(\Sigma) \cap \mathcal{RL}$; that is, those residuated lattices that also model the equations of Σ.

Definition 6.1. We let $\mathcal{RL}^{\mathcal{C}} = \mathrm{HSP}(\mathcal{C})$ denote the subvariety of $\mathcal{RL}$ generated by $\mathcal{C}$, the class of all residuated chains.

In the first subsection, we find an equational basis for $\mathcal{RL}^{\mathcal{C}}$. It follows from Jónsson's Theorem on congruence-distributive varieties (see [31]) that the collection of all subdirectly irreducible algebras of $\mathcal{RL}^{\mathcal{C}}$ is precisely the class $\mathcal{K}$. It is this fact that aids us in discovering a concise basis of just two additional equations for $\mathcal{RL}^{\mathcal{C}}$, our main result of this section. It is easy to make a list of equations that hold in $\mathcal{RL}^{\mathcal{C}}$ since any equation satisfied by chains — for example the distributive law — must hold throughout the subvariety. But to characterize $\mathcal{RL}^{\mathcal{C}}$, we seek an equation ε that captures the fact that the SI algebras of $\mathcal{RL}^{\mathcal{C}}$ are chains. In other words, we need an equation ε such that for any subdirectly irreducible member $\mathbf{L}$ of $\mathcal{RL}$, if $\mathbf{L} \models \varepsilon$ then $\mathbf{L}$ is a chain. In Lemma 6.3, we see that it suffices to capture the join-primeness of $\mathbf{e}$ in $\mathbf{L}$ (which of course must hold in any chain). But given two elements $a, b \in L$ such that $a \vee b = \mathbf{e}$, we are led to investigate the two principal normal submonoids $\mathcal{N}(a)^{-}$ and $\mathcal{N}(b)^{-}$. Any element of their intersection must simultaneousy lie above a product of conjugates of a and a product of conjugates of b and hence the join of these two products. But if $\mathbf{L}$ is to be a chain, this intersection must be trivial, and thus our first approximation for ε becomes something of the form:

$$[\gamma_1(a)\gamma_2(a)\ldots\gamma_j(a)] \vee [\gamma'_1(b)\gamma'_2(b)\ldots\gamma'_k(b)] = \mathbf{e}\,.$$

Other lemmas allow us to unravel the iterated conjugations and, by replacing a and b with $a/(a \vee b)$ and $b/(a \vee b)$, we capture the hypothesis that $a \vee b = \mathbf{e}$ producing finally the four-variable equation

$$\varepsilon : \qquad \lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) = \mathbf{e}\,.$$

It is easy to see that $\mathcal{RL}^{\mathcal{C}} \models \varepsilon$ and, including a weakened form of distributivity

$$\varepsilon_d : \qquad \mathbf{e} \wedge (x \vee y) = (\mathbf{e} \wedge x) \vee (\mathbf{e} \wedge y)\,,$$

we will show that ε and ε_d together define $\mathcal{RL}^{\mathcal{C}}$ relative to $\mathcal{RL}$.

Of course the dual version of ε (in which right division is replaced by left division) could have been used in place of ε. We note this in the second subsection where we also look at some additional equations of interest that hold in $\mathcal{RL}^{\mathcal{C}}$. Finally, in the third subsection, we show that each member $\mathbf{L}$ of $\mathcal{RL}^{\mathcal{C}}$ has the property that the compact elements of $\mathrm{Con}(\mathbf{L})$ form a *relatively normal* lattice, a property investigated for lattices in general by Hart, Snodgrass and Tsinakis in [27, 43].

6.1. *An equational basis for $\mathcal{RL^C}$*

The following observation will make some of our proofs more concise: if $\mathbf{L}$ is totally ordered, then for any $a, b \in L$ with say $a \leq b$ we have that $b/a \geq a/a \geq \mathbf{e}$ so that $\mathbf{L}$ satisfies the equation

$$\varepsilon_1 : \qquad (x/y) \vee (y/x) \geq \mathbf{e}\,.$$

Therefore, this equation holds throughout $\mathcal{RL^C}$ and the next lemma shows that it is a consequence of ε.

Lemma 6.2. *Let ε and ε_1 be as defined above. For any $\boldsymbol{L} \in \mathcal{RL}$, if $\boldsymbol{L} \models \varepsilon$ then $\boldsymbol{L} \models \varepsilon_1$.*

Proof. Suppose $\mathbf{L} \in \mathcal{RL}$ and $\mathbf{L} \models \varepsilon$. In particular, when $z = w = \mathbf{e}$ we have

$$\begin{aligned}\mathbf{e} &= [[x/(x \vee y)] \wedge \mathbf{e}] \vee [[y/(x \vee y)] \wedge \mathbf{e}] \\ &= [(x/x) \wedge (x/y) \wedge \mathbf{e}] \vee [(y/x) \wedge (y/y) \wedge \mathbf{e}] \\ &= [\mathbf{e} \wedge (x/y)] \vee [\mathbf{e} \wedge (y/x)] \leq [(x/y) \vee (y/x)] \wedge \mathbf{e} \leq \mathbf{e}\end{aligned}$$

from which it follows that $\mathbf{e} = [(x/y) \vee (y/x)] \wedge \mathbf{e}$. □

The next lemma is immediate.

Lemma 6.3. *If $\boldsymbol{L} \in \mathcal{RL}$, $\boldsymbol{L} \models \varepsilon_1$ and if $\mathbf{e}$ is join-prime in $\boldsymbol{L}$, then $\boldsymbol{L}$ is a chain.*

We must now show that for subdirectly irreducible members of $\mathcal{RL}$, equation ε implies the join-primeness of $\mathbf{e}$. The next two lemmas will be useful in this endeavor. Lemma 6.4 is an obvious generalization of [5, Theorem 3, p. 324].

Lemma 6.4. *Let $\boldsymbol{L}$ be any residuated lattice and $\{a_i | 1 \leq i \leq n\}, \{b_j | 1 \leq j \leq m\} \subseteq L^-$ finite subsets of the negative cone of $\boldsymbol{L}$ with the property that $a_i \vee b_j = \mathbf{e}$ for any i and j. Then $a \vee b = \mathbf{e}$, where $a = \prod_{i=1}^n a_i$ and $b = \prod_{j=1}^m b_j$.*

Proof. We first fix an arbitrary j and proceed by induction on n to show that $a \vee b_j = \mathbf{e}$. Since this holds for all j, the lemma will then follow by reversing the roles of the a's and b's.

If $n = 1$, the conclusion is immediate. Suppose the result holds for some n and that $\{a_i | \mathbf{e} \leq i \leq (n+1)\}$ together with $\{b_j\}$ satisfy the hypotheses of the lemma. Set $a' := \prod_{i=1}^n a_i$ and $a := \prod_{i=1}^{n+1} a_i$; by the induction hypothesis, $a' \vee b_j = \mathbf{e}$. But now we have

$$a \vee b_j = a' a_{n+1} \vee b_j \geq a' a_{n+1} \vee b_j a_{n+1} = (a' \vee b_j) a_{n+1} = a_{n+1}$$

and of course $a \vee b_j \geq b_j$ so that

$$\mathbf{e} \geq a \vee b_j \geq a_{n+1} \vee b_j = \mathbf{e}$$

which gives the desired result. □

Lemma 6.5. *Suppose $\boldsymbol{L}$ is a residuated lattice such that $\boldsymbol{L} \models \varepsilon$. For all $a, b \in L^{-}$ and for any iterated conjugation maps γ_1, γ_2, if $a \vee b = \mathbf{e}$ then $\gamma_1(a) \vee \gamma_2(b) = \mathbf{e}$.*

Proof. Let $a, b \in L^{-}$ and suppose $a \vee b = \mathbf{e}$. Notice that it suffices to show only that $\gamma(a) \vee b = \mathbf{e}$ for all $\gamma \in \Gamma$ since the same argument applied to $\gamma_1(a)$ and b will yield the final claim. Thus let $\gamma \in \Gamma$ be arbitrary and we proceed by induction on the complexity of γ. If $\gamma = \lambda_c$ for some $c \in L$, then since $\mathbf{L} \models \varepsilon$ we have

$$\gamma(a) \vee b = \lambda_c(a) \vee \rho_{\mathbf{e}}(b) = \lambda_c(a/a \vee b) \vee \rho_{\mathbf{e}}(b/a \vee b) = \mathbf{e}\,,$$

and similarly if $\gamma = \rho_d$ for some $d \in L$. Now suppose the claim holds for some γ; then for any $c, d \in L$, and setting $a' = \gamma(a)$, we have $a' \vee b = \mathbf{e}$ and the same argument as given above shows that

$$[\lambda_c \circ \gamma](a) \vee b = \lambda_c(a') \vee b = \mathbf{e} \text{ and } [\rho_d \circ \gamma](a) \vee b = \rho_d(a') \vee b = \mathbf{e}\,. \qquad \square$$

Finally we are ready to prove the following crucial lemma.

Lemma 6.6. *Suppose $\boldsymbol{L} \in \mathcal{RL}$ is subdirectly irreducible and that $\boldsymbol{L} \models \varepsilon \wedge \varepsilon_d$. Then $\mathbf{e}$ is join-prime in $\boldsymbol{L}$.*

Proof. Equation ε_d implies that $\mathbf{e}$ is join-prime if and only if $\mathbf{e}$ is join-irreducible. So let $a, b \in L$ be such that $a \vee b = \mathbf{e}$. Clearly, $a, b \in L^{-}$ and the two previous lemmas together imply that $\mathcal{N}(a)^{-} \bigcap \mathcal{N}(b)^{-} = \{\mathbf{e}\}$. But then the two corresponding congruences have trivial intersection and since $\mathbf{L}$ is subdirectly irreducible, it must have been that either $a = \mathbf{e}$ or $b = \mathbf{e}$. $\square$

We now have the main theorem of this section:

Theorem 6.7. $\mathcal{RL}^{\mathcal{C}} = \mathrm{Mod}_{\mathcal{RL}}(\varepsilon \wedge \varepsilon_d)$ *where ε and ε_d are the equations*

$$\varepsilon : \lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) = \mathbf{e}$$
$$\varepsilon_d : \mathbf{e} \wedge (x \vee y) = (\mathbf{e} \wedge x) \vee (\mathbf{e} \wedge y)\,.$$

6.2. *Other equations of $\mathcal{RL}^{\mathcal{C}}$*

Recall that for an arbitrary residuated lattice $\mathbf{L}$, the division operations preserve meets in the numerator, and convert joins in the denominator into meets (see Lemma 3.2). In $\mathcal{RL}^{\mathcal{C}}$ we also have the order-dual versions of these equations as listed below. For completeness, we include equations ε and ε_d here, together with the multiplicative duals of all the equations. In the propositions that follow, we investigate the relationships among these equations. All of our discussion is assumed to be relative to the equational theory of $\mathcal{RL}$. We note that [48, Theorem 13.1] contains versions of these propositions for the special case in which $\mathbf{e}$ is the top element of the lattice.

Equations 6.8.

$$\varepsilon_d: \quad (x \vee y) \wedge \mathbf{e} = (x \wedge \mathbf{e}) \vee (y \wedge \mathbf{e})$$
$$\varepsilon: \quad \lambda_z[x/(x \vee y)] \vee \rho_w[y/(x \vee y)] = \mathbf{e}$$
$$\varepsilon': \quad \lambda_z[(x \vee y)\backslash x] \vee \rho_w[(x \vee y)\backslash y] = \mathbf{e}$$

$$\varepsilon_1 : (x/y) \vee (y/x) \geq \mathbf{e} \qquad \varepsilon_1' : (y\backslash x) \vee (x\backslash y) \geq \mathbf{e}$$
$$\varepsilon_2 : x/(y \wedge z) = (x/y) \vee (x/z) \quad \varepsilon_2' : (y \wedge z)\backslash x = (y\backslash x) \vee (z\backslash x)$$
$$\varepsilon_3 : (x \vee y)/z = (x/z) \vee (y/z) \quad \varepsilon_3' : z\backslash(x \vee y) = (z\backslash x) \vee (z\backslash y)$$

Proposition 6.9. *Equation ε implies equation ε_1 and equation ε' implies equation ε_1'.*

Proof. The first half of the statement was proved in Lemma 6.2. The primed version is proved similarly. □

Proposition 6.10. *Equations ε_1 and ε_d together imply both ε_2 and ε_3, each of which implies equation ε_1. Thus, in the presence of ε_d, equations ε_1, ε_2, and ε_3 are equivalent. The analogous statement for the primed equations also holds.*

Proof. To see that $\varepsilon_2 \Rightarrow \varepsilon_1$, note that

$$(x/y) \vee (y/x) \geq [(x \wedge y)/y] \vee [(x \wedge y)/x] = (x \wedge y)/(x \wedge y) \geq \mathbf{e}$$

and $\varepsilon_3 \Rightarrow \varepsilon_1$ since

$$(x/y) \vee (y/x) \geq [x/(x \vee y)] \vee [y/(x \vee y)] = (x \vee y)/(x \vee y) \geq \mathbf{e}\,.$$

Next, suppose ε_d and ε_1 hold. Since it is always true that the left-hand side in ε_2 is greater than or equal to the right-hand side, it suffices to show the reverse inequality. To this end, consider the following:

$$\begin{aligned}
[x/(y \wedge z)]\backslash[(x/y) \vee (x/z)] &\geq [(x/(y \wedge z))\backslash(x/y)] \vee [(x/(y \wedge z))\backslash(x/z)] \\
&\geq [(y \wedge z)/y] \vee [(y \wedge z)/z] = [(y/y) \wedge (z/y)] \vee [(y/z) \wedge (z/z)] \\
&\geq [\mathbf{e} \wedge (z/y)] \vee [\mathbf{e} \wedge (y/z)] = \mathbf{e} \wedge [(z/y) \vee (y/z)] = \mathbf{e}\,.
\end{aligned}$$

The inequality in the second line is from item 8 of Lemma 3.2. We used ε_d and ε_1 in the equalities of the last line. But now we have shown that

$$\mathbf{e} \leq [x/(y \wedge z)]\backslash[(x/y) \vee (x/z)]$$

which is equivalent to ε_2. A similar observation yields ε_3:

$$\begin{aligned}
&[(x/z) \vee (y/z)]/[(x \vee y)/z] \\
&\quad \geq \ [(x/z)/((x \vee y)/z)] \vee [(y/z)/((x \vee y)/z)]
\end{aligned}$$

$$= [x/(x \vee y/z)z] \vee [y/(x \vee y/z)z] \geq [x/x \vee y] \vee [y/x \vee y]$$

$$= [(x/x) \wedge (x/y)] \vee [(y/x) \wedge (y/y)] \geq [\mathbf{e} \wedge (x/y)] \vee [\mathbf{e} \wedge (y/x)]$$

$$= \mathbf{e} \wedge [(x/y) \vee (y/x)] = \mathbf{e}$$

and ε_3 follows. $\square$

Proposition 6.11. *In the presence of equation ε_d, equations ε and ε' are equivalent.*

Proof. The proof in Theorem 6.7 that $\{\varepsilon_d, \varepsilon\}$ forms a basis for $\mathcal{RL}^{\mathcal{C}}$ is easily modified to show that $\{\varepsilon_d, \varepsilon\}$ is also a basis. The proposition now follows. $\square$

Corollary 6.12. *Equation ε_d, together with either of ε or ε', imply all of the others in the list of Equations* 6.8.

6.3. *Congruences in $\mathcal{RL}^{\mathcal{C}}$*

For any $\mathbf{L} \in \mathcal{RL}$ and $a, b \in L$ we always have $\mathcal{N}(a \vee b) \subseteq \mathcal{N}(a) \vee \mathcal{N}(b)$ and $\mathcal{N}(a \wedge b) \subseteq \mathcal{N}(a) \vee \mathcal{N}(b)$. However, if a and b come from the negative cone then we can say more.

Proposition 6.13. *Let $\boldsymbol{L} \in \mathcal{RL}$ and $a, b \in L^{-}$ be arbitrary. Then,*

1. $\mathcal{N}(a \wedge b) = \mathcal{N}(a) \vee \mathcal{N}(b)$.
2. $\mathcal{N}(a \vee b) \subseteq \mathcal{N}(a) \cap \mathcal{N}(b)$. *Equality holds if $\boldsymbol{L} \in SP(\mathcal{C})$ (in particular, equality holds if $\boldsymbol{L} \in \mathcal{RL}^{\mathcal{C}}$).*

Proof. Statement 1 and the inclusion of statement 2 are easy to verify. Suppose now that $a, b \in L^{-}$. If $\mathbf{L}$ is a chain then it is clear that we have equality in statement 2. Suppose $\mathbf{L} \leq \prod_t \mathbf{C}_t$ where $\mathbf{C}_t \in \mathcal{C}$ for all t, and let $x \in [\mathcal{N}(a) \cap \mathcal{N}(b)]^{-}$. Then there are iterated conjugation maps $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $\delta_1, \ldots, \delta_m \in \Gamma$ so that

$$\prod_{j=1}^{n} \gamma_j(a) \leq x \leq \mathbf{e} \text{ and } \prod_{i=1}^{n} \delta_i(b) \leq x \leq \mathbf{e}\,.$$

Fix an arbitrary t and suppose $p_t(a) \geq p_t(b)$ where p_t is the usual projection map. Then

$$\mathbf{e} \geq p_t(x) \geq p_t\left(\prod_{j=1}^{n} \gamma_j(a)\right) = \prod_{j=1}^{n} p_t(\gamma_j(a)) = \prod_{j=1}^{n} p_t(\gamma_j(a \vee b))$$

$$\geq \left[\prod_{j=1}^{n} p_t(\gamma_j(a \vee b))\right]\left[\prod_{i=1}^{m} p_t(\delta_i(a \vee b))\right]$$

$$= p_t\left(\left[\prod_{j=1}^{n} (\gamma_j(a \vee b))\right]\left[\prod_{i=1}^{m} (\delta_i(a \vee b))\right]\right),$$

and an analogous argument shows that

$$\mathbf{e} \geq p_t(x) \geq p_t\left(\left[\prod_{j=1}^{n}(\gamma_j(a \vee b))\right]\left[\prod_{i=1}^{m}(\delta_i(a \vee b))\right]\right)$$

also holds if $p_t(a) \leq p_t(b)$ so that

$$\mathbf{e} \geq x \geq \left[\prod_{j=1}^{n}(\gamma_j(a \vee b))\right]\left[\prod_{i=1}^{m}(\delta_i(a \vee b))\right],$$

which implies that $x \in \mathcal{N}(a \vee b)$. Since a convex normal subalgebra is completely determined by its negative cone, the lemma is proved. □

Corollary 6.14. *For $\boldsymbol{L} \in \mathcal{RL}^{\mathcal{C}}$, the compact members of $\mathcal{CN}(\boldsymbol{L})$ are the principal, convex normal subalgebras $\mathcal{N}(a)$ for $a \in L^{-}$.*

Definition 6.15. A poset, $\mathbf{P}$, is called a **root system** if every principal up-set, $\uparrow p := \{x \in P | x \geq p\}$, is a chain.

Definition 6.16. A lower-bounded, distributive lattice $\mathbf{L}$ is said to be **relatively normal** if its prime ideals form a root-system under set inclusion.

In [27], the following alternative characterization of relatively normal lattices, due to Monteiro [37], is stated:

Theorem 6.17. *Let $\boldsymbol{L}$ be a lower-bounded, distributive lattice. Then the following are equivalent:*

1. *$\boldsymbol{L}$ is relatively normal.*
2. *For every $a, b \in L$ there exist $a', b' \in L$ so that $a' \wedge b' = 0$ and $a \vee b' = a' \vee b = a \vee b$ (it necessarily follows that $a' \leq a$ and $b' \leq b$).*

Proposition 6.18. *If $\boldsymbol{L} \in \mathcal{RL}^{\mathcal{C}}$, then the compact members of* Con($\boldsymbol{L}$) *form a relatively normal lattice.*

Proof. Suppose $a, b \in L^{-}$. Set $a' := (a/b) \wedge \mathbf{e}$ and $b' := (b/a) \wedge \mathbf{e}$. Then notice that

$$a' \vee b' = [(a/b) \wedge \mathbf{e}] \vee [(b/a) \wedge \mathbf{e}] = [(a/b) \vee (b/a)] \wedge \mathbf{e} = \mathbf{e}$$

so that we have

$$\mathcal{N}(a') \wedge \mathcal{N}(b') = \mathcal{N}(a' \vee b') = \mathcal{N}(\mathbf{e}) = \{\mathbf{e}\}.$$

Next, notice that, since $a \leq \mathbf{e}$, $b/a \geq b$ whence $b' = (b/a) \wedge \mathbf{e} \geq b \wedge \mathbf{e} = b$. Thus, $a \wedge b' \geq a \wedge b$ from which it follows that

$$\mathcal{N}(a) \vee \mathcal{N}(b') = \mathcal{N}(a \wedge b') \subseteq \mathcal{N}(a \wedge b).$$

We need to show the reverse inclusion. To this end, observe that

$$(a \wedge b')^2 \leq (a \wedge b')a \leq a^2 \wedge b'a \leq a \wedge [(b/a)a] \leq a \wedge b \leq \mathbf{e}$$

so that $a \wedge b \in \mathcal{N}(a \wedge b')$. The proposition now follows from the symmetry of the definitions. □

7. Concluding Remarks

7.1. *Looking back*

Historically, the origins of residuation theory lie in the study of ideal lattices of rings, and among the first papers published on the subject are those of Ward and Dilworth in the late 1930s (see [15, 16, 45–48]). Over the years, substantial work in this area led to the development of *Multiplicative Ideal Theory* (see [20] by Gilmer or [34] by Larsen and McCarthy).

As for residuation, the closely-related concept of adjunctions was developed as a part of category theory beginning in the 1940s, but it was not until the late 1940s and early 1950s that the idea of a *residuated map* as a separate entity began to appear in papers such as [4, 39]. During the next two decades few works dealt specifically with the subject although several refer to it indirectly or in passing. Some notable examples include the 1963 book by Fuchs [17], particularly his chapter on "Lattice-ordered semigroups", and the 1967 edition of Birkhoff's classic text *Lattice Theory* [5]. The latter addresses the topic in two newly-added sections, "Residuation" and "Applications", which expand on the brief comments found in earlier editions. Finally, in 1972, Blyth and Janowitz published a large tome titled *Residuation Theory*, [7], which was, as they state in the preface, "the first unified account of this topic". Included in their book is a much more extensive bibliography than we give here. As was mentioned earlier, there has been substantial research regarding some specific classes of residuated structures, including lattice-ordered groups and MV-algebras. The theory of lattice-ordered groups is a natural extension of the theory of Riesz spaces (see [1] and the references therein). MV-algebras, introduced by C.C. Chang in 1958 as the algebraic counterparts of $\aleph_0$-valued propositional calculus, also serve as the algebraic structures of truth values for several calculi including fuzzy logics (see [26] and [38]). A comprehensive development of the theory of MV-algebras can be found in [9].

7.2. *Looking ahead*

Recently, Hart, Rafter and Tsinakis [28] began an investigation into the general structure of commutative residuated objects and a large part of this paper has been devoted to extending their results to the non-commutative case. These works represent an attempt to understand such structures in a comprehensive way and from the viewpoint of universal algebra. Here, in particular, we develop the concept of a normal subalgebra and we give a canonical description of the elements of the normal subalgebra generated by an arbitrary subset. This, in turn, allows us to completely describe the connection between the subalgebra lattice and the lattice of congruences, showing that $\mathcal{RL}$ is an "ideal variety". Furthermore, one always likes to have a concise equational characterization for a variety and we provide that here for both $\mathcal{RL}$ and $\mathcal{RL}^{\mathcal{C}}$. In the process, we show that the members of $\mathcal{RL}^{\mathcal{C}}$ have certain properties that we believe could lead to new decomposition theorems for the finite objects of this class. Such results should further illuminate their structure in a fundamental way.

We believe that the subject of residuated lattices is still wide open with many areas ripe for possible research. For example, several new results regarding the atoms of the lattice of subvarieties of $\mathcal{RL}$ have been obtained by the participants of the senior author's (Tsinakis') Spring 2000 and Fall 2001 seminars on residuation theory. These, together with research into problems of decidability and free objects, are currently being prepared for publication [2, 18].

In this section we outline a few questions and some possible lines of research, several of which we hope to investigate in the near future. We hesitate to call them "open questions" since this phrase tends to imply that they have already resisted efforts to solve them. Rather, these are based on marginal notes made during our research and as yet we have spent little time working on them. Perhaps some will turn out to be truly challenging while others may yield quickly once attention has been focused on them.

- Does there exist a representation theorem for the class $\mathcal{RL}$ or the class $\mathcal{RL}^{\mathcal{C}}$? It is well-known that any group can be represented as a group of permutations of a set (Cayley's representation theorem) and any l-group as a group of order automorphisms of a totally ordered set (Holland's representation theorem [29]). We wonder whether it is possible that each distributive member of $\mathcal{RL}$, or perhaps some suitable subclass, can be embedded into the lattice of residuated self-maps of some chain (with composition of maps as the multiplication). In [6] we point out that the obvious embedding fails in general.
- Is there a "nice" characterization of those lattices that admit residuation? We know, for example (see [6]), that any finite lattice admits residuation as does any upper-bounded chain while any lower bounded lattice without a top element cannot be residuated. Perhaps a starting point would be to determine whether (or which) unbounded chains admit residuation.
- Can one fruitfully explore further the lattice of subvarieties of $\mathcal{RL}$, denoted $\mathbf{L}(\mathcal{RL})$, and perhaps shed some light on parts of its structure (see Fig. 1 in the introduction)? Much work has been done along these lines for various subclasses. In particular, much is known about the intervals below $\mathcal{L}G$, ℓ-groups, and below $\mathcal{B}r$, Brouwerian algebras. For example, it is known that the variety of lattice-ordered Abelian groups is an atom in the subvariety lattice of ℓ-groups, and hence also in $\mathbf{L}(\mathcal{RL})$, and that this atom has uncountably many covers (see [30] and [40]). It would be of interest if one could further illuminate the structure near the bottom of $\mathbf{L}(\mathcal{RL})$, perhaps by describing some interesting classes of atoms. Recent work ([2, 18]) has made some progress in this direction.
- Since the variety $\mathcal{RL}$ is an ideal variety, we can define the so-called Mal'cev product on the lattice of subvarieties in the following way: given two subvarieties $\mathcal{V}_1$ and $\mathcal{V}_2$, define

$$\mathcal{V}_1 * \mathcal{V}_2 := \{\mathbf{L} \in \mathcal{RL} | \exists \mathbf{H} \in \mathrm{Sub}_{\mathcal{CN}}(\mathbf{L}) \text{ with } \mathbf{H} \in \mathcal{V}_1 \text{ and } \mathbf{L}/\theta_{\mathbf{H}} \in \mathcal{V}_2\}\,.$$

Of great interest would be any results that contribute to an understanding of the multiplicative structure of $\mathbf{L}(\mathcal{RL})$ with respect to this operation.

- Let X be any finite non-empty chain and let us order the free monoid, $\mathcal{F}_{\mathcal{M}}(X)$, in the following way: for two words $w_1, w_2 \in \mathcal{F}_{\mathcal{M}}(X)$ we define $w_1 < w_2$ if and only if either $\mathit{length}(w_2) < \mathit{length}(w_1)$, or $\mathit{length}(w_1) = \mathit{length}(w_2)$ and w_1 precedes w_2 in the dual lexicographic ordering induced by the ordering on X. This ordering is a compatible residuated total order on $\mathcal{F}_{\mathcal{M}}(X)$ and there exist many other total orderings with respect to which the free monoid is residuated. Free monoids are of course cancellative and it is known that the members of $\mathcal{RL}$ that satisfy the cancellative property form a subvariety (in this setting, the cancellative property is captured by the equations $(xy)/y = x = y\backslash(yx)$). Denoting this subvariety $\mathcal{C}an\mathcal{RL}$, one can show that $\mathcal{C}an\mathcal{RL}^{\mathcal{C}} := \mathcal{C}an\mathcal{RL} \cap \mathcal{RL}^{\mathcal{C}}$ is generated by residuated totally ordered, free monoids. It would be of interest to provide a "canonical" description of the free algebras of $\mathcal{C}an\mathcal{RL}^{\mathcal{C}}$, in the style of the description of the free objects in the variety of representable lattice-ordered groups (see, for example, [41]).
- Can one describe all residuated total orders on $\mathcal{F}_{\mathcal{M}}(X)$, where X is a finite set?
- Let $\mathbf{Z}$ denote the integers (with the usual ordering) and $\mathbf{Z}^-$ its negative cone. Under the usual addition, these two chains become members of $\mathcal{RL}$. It is well known that $\mathbf{Z}$ generates the variety of lattice-ordered abelian groups, which is an atom in the subvariety lattice $\mathbf{L}(\mathcal{RL})$. It is also simple to see that that $\mathbf{Z}^-$ also generates an atom in the subvariety lattice. It is shown in [2] that these are the *only* two atoms that lie below the subvariety of commutative, cancellative residuated lattices — but it is an open question whether there are any other atoms below $\mathcal{C}an\mathcal{RL}$ itself.
- Although in many well-known subvarieties of $\mathcal{RL}$ distributivity is a consequence of the cancellative law, this is not true in general (see [2]). We would like to know to what extent this implication fails. In particular, is every finite lattice a sublattice of some cancellative, residuated lattice?

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