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THE UNIMODALITY OF PURE O-SEQUENCES OF TYPE TWO IN FOUR VARIABLES

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ABSTRACT. Since the 1970's, great interest has been taken in the study of pure O-sequences, which, due to Macaulay's theory of inverse systems, have a bijective correspondence to the Hilbert functions of Artinian level monomial algebras. Much progress has been made in classifying these according to their shape. Macaulay's theorem immediately gives us that all Artinian algebras in two variables have unimodal Hilbert functions. Furthermore, it has been shown that all Artinian level monomial algebras of type two in three variables have unimodal Hilbert functions. This paper will classify all Artinian level monomial algebras of type two in four variables into two classes of ideals, prove that they are strictly unimodal and show that one of the classes is licci.

1. Introduction. The study of pure O-sequences began in 1977 with the work of Stanley [**19**]. In the relatively short time since then, these objects have appeared in the study of a wide array of other mathematical areas, some much older than pure O-sequences themselves. In his initial study of pure O-sequences, Stanley conjectured that the h-vector of a matroid complex is a pure O-sequence [**19**]. Since then, many mathematicians have taken an interest in studying pure O-sequences. Although Stanley's conjecture is still open, there have been a number of interesting, albeit partial, results. These results include proving the conjecture for paving matroids [**14**], cotransversal matroids [**16**], one-dimensional matroid complexes [**21**] and lattice path matroids [**18**], among others. Additional connections have been found between pure O-sequences and the areas of topological combina-

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torics **[3]**, error-correcting codes **[12]** and more; we refer the reader to **[1]** for other examples.

One of the promising approaches for addressing pure O-sequences is to study them in light of their bijective correspondence to the Hilbert functions of Artinian level monomial algebras, which is what we will do in this paper. This bijective correspondence between these objects is due to Macaulay's theory of inverse systems. More background on inverse systems and Artinian level monomial algebras will be given in the next section. Here we will recall some definitions and previous results to motivate this paper.

An *order ideal* is a non-empty set X of (monic) monomials such that, if $M \in X$ and N is a monomial dividing M, then $N \in X$. The *h-vector* of X is the sequence $\underline{h}= (h_0 = 1, h_1, \ldots, h_e)$ (with $h_e \neq 0$) which counts the monomials of X in each degree. An order ideal is *pure* if all maximal monomials of X have the same degree. A *pure O-sequence* is the h-vector of a pure order ideal. The *type* of an O-sequence is the number of maximal monomials (ordered by divisibility) in the order ideal. A sequence is *unimodal* if it does not increase after a strict decrease; a sequence is *strictly unimodal* if it is unimodal and only constant in its peak degree(s). We should note that pure O-sequences and the Hilbert functions of algebras are not affected by the characteristic; therefore, without loss of generality, we assume characteristic zero throughout this paper.

Example 1.1. Let X be a pure order ideal in three variables and

$$\{x^2y^2, z^4, x^3z\} \in X.$$

Then we must also have

$$\{x, y, z, x^2, y^2, z^2, xy, xz, x^3, z^3, xy^2, x^2y, x^2z\} \in X.$$

This gives us that $\underline{h} = (1, 3, 5, 5, 3)$. By design, the order ideal is pure since the maximal monomials, (x^2y^2, z^4, x^3z), all have the same degree; furthermore, we have that the type is three since there are three generating monomials. This h-vector is strictly unimodal.

Recently, there has been interest and progress in classifying and characterizing the shape of pure O-sequences. In particular, two main results are due to Hibi and to Hausel. In **[9,** Theorem 1.1], Hibi showed that all pure O-sequences $\underline{h} = (1, h_1, h_2, \ldots, h_e)$ are *flawless* or, equivalently, that

$$h_i \leq h_{e-i} \quad \text{for all } i \leq \left\lfloor \frac{e}{2} \right\rfloor .$$

Hibi also showed that

$$h_{i-1} \leq h_i \quad \text{for all } i \leq \left\lfloor \frac{e}{2} \right\rfloor ,$$

so the "first half" of $\underline{h}$ is non-decreasing. Hausel built on this result ([**7,** Theorem 6.3]) showing that, in addition to the first half being non-decreasing, it is differentiable. Thus, the first difference of the first half of $\underline{h}$ satisfies Macaulay's theorem, which implies that it is the h-vector for an order ideal (which is not necessarily pure). The converse of this was proven by Boij, et al. in [**1**]. Specifically, they showed that a finite non-decreasing O-sequence $\underline{h}$ is the "first half" of a pure O-sequence if and only if it is differentiable.

These results give us a lot of information about the first half of the h-vector, which naturally leads one to ask about the second half of the h-vector, in particular whether the h-vector, or equivalently the Hilbert function of an Artinian level monomial algebra, is unimodal. There have been several results which give families where the Hilbert function of Artinian level monomial algebras are unimodal. In two variables, Macaulay's maximal growth theorem immediately implies that all Artinian algebras have unimodal Hilbert functions. In more variables, one tool that has been useful is the weak Lefschetz property (WLP). This property says that multiplication by a general linear form has maximal rank from any component of the algebra to the next. A consequence of this property is that the Hilbert function of the algebra is unimodal. This is due to the standard grading of the algebra which was first noted in [**6**]. Using this tool, Stanley [**20**], Watanabe [**22**] and Reid, Roberts, and Roitman [**17**] showed that all monomial complete intersections have unimodal Hilbert functions. Furthermore, in [**1,** Corollary 6.8], the authors showed that all Artinian level monomial algebras of type two in three variables have the WLP in characteristic zero and thus have unimodal Hilbert functions in any characteristic. Unfortunately, [**1,** Theorem 7.17] shows that, in three or more variables, the only cases where the weak Lefschetz property is guaranteed for level Artinian monomial algebras in characteristic zero are type one for any number of variables and type two in three variables.

The first counterexample of a codimesion three level Artinian monomial algebra which fails to have the WLP (with type as low as three) was found by Zanello [**23**]. Furthermore, in three variables, Brenner and Kaid ([**2**]) showed that WLP can fail for a level type three monomial algebra, even an almost complete intersection.

Even without the WLP holding for these algebras, one can still ask if pure O-sequences are unimodal. However, for high enough type, the unimodality of pure O-sequences can fail, as several examples have been found. The first example of a non-unimodal pure O-sequence is due to Stanley [**19**]. He showed that $(1, 505, 2065, 3395, 3325, 3493)$ is a pure O-sequence, but clearly it is not unimodal. Furthermore in [**1,** Theorem 3.9], the authors found an infinite family of pure O-sequences which are non-unimodal. In particular, they found that if M is any positive integer and $r \geq 3$, a fixed integer, then there exists a pure O-sequence in r variables which is non-unimodal, having exactly M maxima. It is clear from these last examples that not all pure O-sequences will be unimodal, but it is still interesting to ask if fixing the type to be low enough will guarantee that a pure O-sequence is unimodal.

It follows from what we have said above that pure O-sequences of type one in any number of variables, and type two in three variables, are known to be unimodal. In this paper, we will settle the next open case. Specifically, we will show that any pure O-sequence of type two in four variables is strictly unimodal. Since the WLP does not necessarily hold in this case, we will approach this problem using modified techniques. In particular, we will rely heavily on the fact that the Hilbert functions of complete intersections peak in the middle degree. We will also explore the liaison classes of these algebras.

2. Background. In this section, we will review some definitions and results that are needed in the paper.

Let $R = k[x_1, \ldots, x_r]$ where k is a field of characteristic zero. Let I be a monomial ideal of R with no non-zero elements of degree 1. We will consider a standard graded Artinian monomial k-algebra with codimension r,

$$R/I = \bigoplus_{i \geq 0} (R/I)_i.$$

The Hilbert function of R/I is

$$H(R/I,i) = \dim_k (R/I)_i = \dim_k R_i - \dim_k I_i.$$

Let $\mathfrak{m} = (x_1, \ldots, x_r)$ be the maximal ideal of R; the homogeneous maximal ideal in R/I is $\overline{\mathfrak{m}} = (\overline{x_1}, \ldots, \overline{x_r})$. The *socle* of R/I is the annihilator of $\overline{\mathfrak{m}}$, so $\operatorname{soc}(R/I) = \{a \in R/I \mid a\overline{\mathfrak{m}} = 0\}$. The Hilbert function of an Artinian algebra is finite, and thus $H(R/I) = (h_0 = 1, h_1, h_2, \ldots, h_e)$, where e is the last degree i for which $H(R/I,i) \neq 0$ ($h_i > 0$ for $0 \leq i \leq e$). Since our algebras are level, the socle is necessarily concentrated in degree e, called the *socle degree*. The *type* of R/I is $\dim(\operatorname{soc}(R/I)) = h_e$. We notice that some of this notation is similar to that of the pure order ideal because these objects are actually the same due to the theory of inverse systems.

Macaulay developed the theory of inverse systems, which helps translate between order ideals and Artinian algebras. To set up this theory, let

$$R = k[x_1, \ldots, x_r]$$

and

$$S = k[y_1, \ldots, y_r].$$

S acts on R in the following way: if $F \in R$, then

$$y_i \circ F = \left(\frac{\partial}{\partial x_i}\right) F.$$

For inverse systems, we can assume that $\operatorname{char}(k)$ equals 0 and thus essentially ignore the coefficients. This is to ensure the correct Hilbert function which is not affected by the characteristic. This also ensures that the action results in submodules that are consistent with pure O-sequences. Using the language of inverse systems, the elements produced by this action are called *derivatives*, even though they differ slightly from the traditional concept of derivatives where the coefficients are retained. There is a one-to-one correspondence between ideals of S and S-submodules of R given by the function

$$\varphi_1 : \{\text{ideals of } S\} \longrightarrow \{S\text{-submodules of } R\}$$
$$\varphi_1(I) = \{F \in R \mid G \circ F = 0 \text{ for all } G \in I\}.$$

We denote $\varphi_1(I)$ as $I^{\perp}$ and call it the *inverse system* to I.

In this paper, we will use the inverse of φ_1, which is

$$\varphi_2(M) = \mathrm{ann}_S(M).$$

When focusing on monomial ideals (as we are in this paper), S can be thought of as being the same polynomial ring as R. Furthermore, in this case, we have that in any degree d, the inverse system $I_d^\perp$ is spanned by all the monomials in R_d that are not in I_d. We refer the reader to [**4, 5**] or [**11,** Appendix] for more on inverse systems.

This correspondence also preserves certain properties of the order ideal and Artinian monomial algebras. We have that the order ideal is pure if and only if the corresponding Artinian monomial algebra is level. Furthermore, the type of the order ideal is the same as the type of the algebra. Thus, we have that Artinian level monomial algebras and their Hilbert functions have a bijective correspondence with pure order ideals and pure O-sequences.

For this paper, we will focus on Artinian level monomial algebras of type two in four variables. Thus, we can derive a lot of information about these algebras from this theory of inverse systems. Let $R = k[x,y,z,w]$. If A is an Artinian level monomial algebra of type two in four variables, then A can be thought of as the inverse system of two monomials of the same degree. Thus, A is isomorphic to $R/(J \cap K)$ where $J = (x^{a_1}, y^{a_2}, z^{a_3}, w^{a_4})$ and $K = (x^{b_1}, y^{b_2}, z^{b_3}, w^{b_4})$ such that $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$. Furthermore, one can derive the Hilbert series of this algebra from the following exact sequence:

$$0 \longrightarrow R/(J \cap K) \longrightarrow R/J \oplus R/K \longrightarrow R/(J+K) \longrightarrow 0.$$

This gives that

$$\begin{aligned}\mathrm{Hilb}\,(R/(J\cap K),t) &= \mathrm{Hilb}\,(R/J,t) + \mathrm{Hilb}\,(R/K,t) \\ &\quad - \mathrm{Hilb}\,(R/(J+K),t) \\ &= \frac{\prod_{i=1}^4 (1-t^{a_i})}{(1-t)^4} + \frac{\prod_{i=1}^4 (1-t^{b_i})}{(1-t)^4} \\ &\quad - \frac{\prod_{i=1}^4 (1-t^{c_i})}{(1-t)^4}\end{aligned}$$

where $c_i = \min\{a_i, b_i\}$ for all $i = 1, 2, 3, 4$.

Additional tools used in this paper come from liaison theory. Two ideals A and B are CI-linked (respectively, G-linked) if there exists a complete intersection ideal C (respectfully, Gorenstein ideal) such that $C \subseteq A \cap B$, $[C : A] = B$ and $[C : B] = A$. This is denoted as $A \overset{C}{\sim} B$. Two ideals are in the same liaison class if they can be linked together in a finite number of links. An ideal is *licci* if it is in the *li*aison *c*lass of a *c*omplete *i*ntersection, where all the links are complete intersections. An ideal is *glicci* if it is the liaison class of a complete intersection, with all Gorenstein ideal links.

The liaison classes of ideals are interesting to study since linkage preserves several invariants such as codimension, certain cohomology modules and the property of being arithmetically Cohen-Macaulay. The study of liaison theory has led to the construction of particular ideals through basic double linkage. Let

$$J \subset I \subset R = k[x_1, \ldots, x_r],$$

where J and I are homogeneous ideals with $\operatorname{codim}(J) = \operatorname{codim}(I) - 1$. Let $f \in R$ be homogeneous, with $J : f = J$. Then $I' := f \cdot I + J$ is a *basic double link*. This name comes from the fact that if I is unmixed and R/J is Cohen-Macaulay and generically Gorenstein, then I' can be Gorenstein linked to I in two steps. There has been much progress studying liaison, but we will only state what we will need in the rest of the paper, namely, the Hilbert function formula of a basic double link. We refer the reader to **[13, 15]** for more information on liaison theory.

For the next lemma and throughout this work we will use the notation $H(a_1, a_2, \ldots, a_r)$. This denotes the Hilbert function of a complete intersection of the form $(x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$.

Lemma 2.1. *Let $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ be a complete intersection in the ring $k[x_1, \ldots, x_r]$. Let ΔH be the first difference of its Hilbert function. Then*

$$\Delta H = H(a_1, a_2, \ldots, a_{r-1}) - H(a_1, a_2, \ldots, a_{r-1})(-a_r),$$

where $H(a_1, a_2, \ldots, a_{r-1})$ and $H(a_1, a_2, \ldots, a_{r-1})(-a_r)$ are in the ring $k[x_1, \ldots, x_{r-1}]$. Any permutation of the a_i is equally valid.

3. Main theorem. We ultimately want to look at the unimodality of the Hilbert functions of Artinian level monomial algebras of type two in four variables. First, we classify these algebras into two cases.

Lemma 3.1. *Let $R = k[x, y, z, w]$ and I be an Artinian monomial ideal such that R/I is level of type two. Then I has one of the following two forms, up to a change of variables. In both cases, we have $a \geq \alpha > 0$, $b \geq \beta > 0$, $c \geq \gamma > 0$ and $d \geq \delta > 0$.*

(i) $(x^a, y^b, z^c, w^d, x^\alpha w^\delta, y^\beta w^\delta, z^\gamma w^\delta)$ *where* $a+b+c+\delta = \alpha+\beta+\gamma+d$ *and* $d > \delta$. *The Hilbert function of* R/I *is*

$$H_{R/I} = H(a, b, c, \delta) + H(\alpha, \beta, \gamma, d - \delta)(-\delta).$$

(ii) $(x^a, y^b, z^c, w^d, x^\alpha z^\gamma, x^\alpha w^\delta, y^\beta z^\gamma, y^\beta w^\delta)$ *where* $\alpha + \beta + c + d = a + b + \gamma + \delta$. *The Hilbert function of* R/I *is*

$$H_{R/I} = H(a - \alpha, b, \gamma, \delta)(-\alpha) + H(\alpha, b - \beta, \gamma, \delta)(-\beta) + H(\alpha, \beta, c, d).$$

Proof. Section 2 gives $R/I \cong R/(J \cap K)$ where $J = (x^{a_1}, y^{a_2}, z^{a_3}, w^{a_4})$ and $K = (x^{b_1}, y^{b_2}, z^{b_3}, w^{b_4})$ with $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$. Up to a change of variables we have two cases:

(i) $a_i \geq b_i$ for three i (if $b_i \geq a_i$ for three i, swap J and K)
(ii) $a_i \geq b_i$ for two i

For case (i) assume, without loss of generality, that $a_i \geq b_i$ for $i \leq 3$ and $a_4 \leq b_4$. Thus, from [**8,** Proposition 1.2.1], we have

$$I := J \cap K = (x^{a_1}, y^{a_2}, z^{a_3}, w^{b_4}, x^{b_1} w^{a_4}, y^{b_2} w^{a_4}, z^{b_3} w^{a_4}),$$

and

$$\begin{aligned} \operatorname{Hilb}(R/I, t) &= \frac{\prod_{i=1}^4 (1 - t^{a_i})}{(1 - t)^4} + \frac{\prod_{i=1}^4 (1 - t^{b_i})}{(1 - t)^4} \\ &\quad - \frac{(1 - t^{a_4}) \prod_{i=1}^3 (1 - t^{b_i})}{(1 - t)^4} \\ &= \frac{\prod_{i=1}^4 (1 - t^{a_i})}{(1 - t)^4} + \frac{t^{a_4} (1 - t^{b_4 - a_4}) \prod_{i=1}^3 (1 - t^{b_i})}{(1 - t)^4}. \end{aligned}$$

For case (ii), assume that $a_i \geq b_i$ for $i = 1, 2$ and $a_j \leq b_j$ for $i = 3, 4$. Thus, from [**8,** Proposition 1.2.1], we have

$$I := J \cap K = (x^{a_1}, y^{a_2}, z^{b_3}, w^{b_4}, x^{b_1} z^{a_3}, x^{b_1} w^{a_4}, y^{b_2} z^{a_3}, y^{b_2} w^{a_4}),$$

and

$$\begin{aligned}
\operatorname{Hilb}(R/I, t) &= \frac{\prod_{i=1}^4 (1 - t^{a_i})}{(1-t)^4} + \frac{\prod_{i=1}^4 (1 - t^{b_i})}{(1-t)^4} \\
&\quad - \frac{(1 - t^{b_1})(1 - t^{b_2})(1 - t^{a_3})(1 - t^{a_4})}{(1-t)^4} \\
&= \frac{\prod_{i=1}^4 (1 - t^{b_i})}{(1-t)^4} + \frac{\prod_{i=1}^4 (1 - t^{a_i})}{(1-t)^4} \\
&\quad - \frac{(1 - t^{b_i}) \prod_{i=2}^4 (1 - t^{a_i})}{(1-t)^4} \\
&\quad + \frac{(1 - t^{b_i}) \prod_{i=2}^4 (1 - t^{a_i})}{(1-t)^4} \\
&\quad - \frac{(1 - t^{b_1})(1 - t^{b_2})(1 - t^{a_3})(1 - t^{a_4})}{(1-t)^4} \\
&= \frac{\prod_{i=1}^4 (1 - t^{b_i})}{(1-t)^4} \\
&\quad + \frac{\prod_{i=3}^4 (1 - t^{a_i})[(1 - t^{a_2})[(1 - t^{a_1}) - (1 - t^{b_1})]]}{(1-t)^4} \\
&\quad + \frac{\prod_{i=3}^4 (1 - t^{a_i})[(1 - t^{b_1})[(1 - t^{a_2}) - (1 - t^{b_2})]]}{(1-t)^4} \\
&= \frac{\prod_{i=1}^4 (1 - t^{b_i})}{(1-t)^4} + \frac{t^{b_1}(1 - t^{a_1 - b_1}) \prod_{i=2}^4 (1 - t^{a_i})}{(1-t)^4} \\
&\quad + \frac{t^{b_2}(1 - t^{b_1})(1 - t^{a_2 - b_2})(1 - t^{a_3})(1 - t^{a_4})}{(1-t)^4}.
\end{aligned}$$

After renaming the variables, we have that the ideals and their Hilbert functions match those of the Proposition. □

Theorem 3.2. *Let $R = k[x, y, z, w]$, and let I be a monomial Artinian ideal such that R/I is level of type two. Then the Hilbert function of R/I is strictly unimodal.*

We will prove this theorem by looking at the two cases given in Lemma 3.1 with Propositions 3.3 and 3.4, respectively.

Proposition 3.3. *Let*

$$I = (x^a, y^b, z^c, w^d, x^\alpha w^\delta, y^\beta w^\delta, z^\gamma w^\delta),$$

where $a+b+c+\delta = \alpha+\beta+\gamma+d$ *with* $a \geq \alpha > 0$, $b \geq \beta > 0$, $c \geq \gamma > 0$, $d > \delta > 0$ *and*

$$H_{R/I} = H(a,b,c,\delta) + H(\alpha,\beta,\gamma,d-\delta)(-\delta) =: L + J.$$

Then, this Hilbert function is strictly unimodal.

Proof. For strict unimodality, it is enough to show that the first difference of $H_{R/I}$ is positive, possibly zero, then negative. First observe that L and J are the Hilbert functions of complete intersections; thus, they are strictly unimodal and peak in the middle degree. We have that the first differences of L and J are positive then possibly zero in degrees less than or equal to

$$A := \left\lfloor \frac{a+b+c+\delta-4}{2} \right\rfloor$$

and

$$B := \left\lfloor \frac{\alpha+\beta+\gamma+d+\delta-4}{2} \right\rfloor,$$

respectively. Furthermore, ΔL and ΔJ are possibly zero then negative in all degrees after A and B, respectively. We note that $A \leq B$ since

$$a+b+c+\delta-4 = \alpha+\beta+\gamma+d-4 < \alpha+\beta+\gamma+d+\delta-4.$$

This gives that both ΔL and ΔJ, and thus $\Delta H_{R/I}$, are positive then possibly zero from degree 0 to A. Similarly, $\Delta H_{R/I}$ is possibly 0 then negative from degree B to the end.

First, we assume that $\Delta H_{R/I} = 0$ for some degree $t \leq A$ or $t \geq B$, then $\Delta L = 0$ and $\Delta J = 0$ in degree t. Since both L and J are strictly unimodal, $\Delta H_{R/I}$ cannot become positive after this degree t. Furthermore, if $\Delta H_{R/I} = 0$ for some degree $t \leq A$ or $t \geq B$, then when $\Delta H_{R/I}$ becomes negative either ΔL or ΔJ is negative and the other equals 0, or both segments are negative. In either case $\Delta H_{R/I}$ will remain negative until the end. Thus, if $\Delta H_{R/I} = 0$ for some $t \leq A$

or $t \geq B$, then $H_{R/I}$ is strictly unimodal. Therefore, without loss of generality, assume that $\Delta H_{R/I} \neq 0$ in degrees less than or equal to A and greater than or equal to B.

To prove the strict unimodality of $H_{R/I}$, it is enough to show that $\Delta^2 H_{R/I} \leq 0$ for all degrees between A and B. If this is true, then $\Delta H_{R/I}$ will be decreasing or constant between A and B. Thus, if $\Delta H_{R/I} = 0$, it will not become positive again; furthermore, once $\Delta H_{R/I} < 0$ it will remain negative until it ends. This would imply that $H_{R/I}$ is strictly unimodal.

To show that $\Delta^2 H_{R/I}(t) \leq 0$ for all degrees $A \leq t \leq B$, we will use Lemma 2.1 to decompose $\Delta H_{R/I}$ as

$$\begin{aligned}\Delta H_{R/I} = {} & H(a,b,c) + H(\alpha,\beta,\gamma)(-\delta) \\ & - H(a,b,c)(-\delta) - H(\alpha,\beta,\gamma)(-d).\end{aligned}$$

Each segment is the Hilbert function of a complete intersection; thus, we have that

$$\Delta H(a,b,c) \leq 0 \text{ after degree } \left\lfloor \frac{a+b+c-3}{2} \right\rfloor =: X$$
$$-\Delta H(\alpha,\beta,\gamma)(-d) \leq 0 \text{ through degree } \left\lfloor \frac{\alpha+\beta+\gamma+2d-3}{2} \right\rfloor =: W.$$

Comparing these degrees to degrees A and B gives

$$\begin{aligned} a+b+c-3 \leq a+b+c+\delta-4 &\Longrightarrow X \leq A \\ &\Longrightarrow \Delta H(a,b,c) \leq 0 \quad \text{for all degrees} \geq A. \\ \alpha+\beta+\gamma+2d-3 \geq \alpha+\beta+\gamma+d+\delta-4 &\Longrightarrow W \geq B \\ &\Longrightarrow -\Delta H(\alpha,\beta,\gamma)(-d) \leq 0 \quad \text{for all degrees} \leq B. \end{aligned}$$

The above inequalities give that $\Delta H(a,b,c)$ and $-\Delta H(\alpha,\beta,\gamma)(-d)$ are non-positive between degrees A and B. This leaves $\Delta[H(\alpha,\beta,\gamma)(-\delta) - H(a,b,c)(-\delta)]$ to check. We note that since $H(\alpha,\beta,\gamma)(-\delta)$ and $H(a,b,c)(-\delta)$ start in the same degree and $a \geq \alpha$, $b \geq \beta$ and $c \geq \gamma$, the difference of these two Hilbert functions will initially be zero and then decrease until at least the peak value of $H(a,b,c)(-\delta)$. We also notice that this peak value is at

$$\left\lfloor \frac{a+b+c+2\delta-3}{2} \right\rfloor,$$

which is greater than or equal to B (since $a+b+c+2\delta-3 = \alpha+\beta+\gamma+d+\delta-3 \geq \alpha+\beta+\gamma+d-4$). Thus, $\Delta[H(\alpha,\beta,\gamma)(-\delta)-H(a,b,c)(-\delta)]$ is non-positive until at least degree B. Therefore, we have that $\Delta^2 H_{R/I}$ is zero or negative between A and B. This implies that $\Delta H_{R/I}$ is positive, possibly zero, then negative so $H_{R/I}$ is strictly unimodal. □

Proposition 3.4. *Let*

$$I = (x^a, y^b, z^c, w^d, x^\alpha z^\gamma, x^\alpha w^\delta, y^\beta z^\gamma, y^\beta w^\delta),$$

where $\alpha+\beta+c+d = a+b+\gamma+\delta$ *with* $a \geq \alpha > 0$, $b \geq \beta > 0$, $c \geq \gamma > 0$, $d \geq \delta > 0$ *and*

$$\begin{aligned} H_{R/I} &= H(a-\alpha,b,\gamma,\delta)(-\alpha) \\ &\quad + H(\alpha,b-\beta,\gamma,\delta)(-\beta) + H(\alpha,\beta,c,d) \\ &=: H_1 + H_2 + H_3. \end{aligned}$$

Then, this Hilbert function is strictly unimodal.

Proof. For this proof, we will assume, without loss of generality, that $d \geq a$, $d \geq b$ and $d \geq c$. If this is not true, then find the variable with the highest exponent and set that variable to be w. It is enough to show that the first difference of $H_{R/I}$ is positive, then possibly zero, then negative. Since all three segments of the decomposition are Hilbert functions of complete intersections, we know that they must be strictly unimodal and peak in the middle degree. Thus, ΔH_1, ΔH_2 and ΔH_3 are positive then possibly zero in all degrees less than or equal to:

$$\begin{aligned} A &:= \left\lfloor \frac{a+\alpha+b+\gamma+\delta-4}{2} \right\rfloor \\ B &:= \left\lfloor \frac{\alpha+b+\beta+\gamma+\delta-4}{2} \right\rfloor \\ C &:= \left\lfloor \frac{\alpha+\beta+c+d-4}{2} \right\rfloor, \end{aligned}$$

respectively. Furthermore, ΔH_1, ΔH_2 and ΔH_3 are possibly zero then negative in all degrees after A, B and C, respectively. Thus, before

$\min\{A,B,C\} =: T$, all three segments are non-negative and, as a result, so is $\Delta H_{R/I}$. Similarly, $\Delta H_{R/I}$ is non-positive in all degrees after $\max\{A,B,C\} =: Q$.

First, assume that $\Delta H_{R/I} = 0$ in a degree $t \leq T$ or $t \geq Q$. In either case, the first differences of H_1, H_2 and H_3 all must equal 0. Since H_1, H_2 and H_3 are strictly unimodal, once $\Delta H_{R/I} = 0$, it cannot become positive again. Furthermore, once $\Delta H_{R/I}$ becomes negative, it will remain negative. Therefore, if $\Delta H_{R/I} = 0$ in a degree $t \leq T$ or $t \geq Q$, then $H_{R/I}$ is strictly unimodal. In particular, this addresses the case where $A = B = C$ (since $T = Q$). Furthermore, without loss of generality, we will assume that $\Delta H_{R/I} \neq 0$ in any degree $t \leq T$ or $t \geq Q$.

To prove the strict unimodality of $H_{R/I}$, it is enough to show that $\Delta^2 H_{R/I} \leq 0$ for all degrees between T and Q. If this is true, then $\Delta H_{R/I}$ will be decreasing or constant between T and Q. Thus, if $\Delta H_{R/I} = 0$, it will not become positive again; furthermore, once $\Delta H_{R/I} < 0$, it will remain negative until it ends. This would imply that $H_{R/I}$ is strictly unimodal.

We will now investigate the relationship between A, B and C. We note that $A \geq C$ since

$$a + \alpha + b + \gamma + \delta - 4 = 2\alpha + \beta + c + d - 4 \geq \alpha + \beta + c + d - 4.$$

This leaves three cases to check.

(1) $Q := \max\{A,B,C\} = B$ and $T := \min\{A,B,C\} = C$;
(2) $Q := \max\{A,B,C\} = A$ and $T := \min\{A,B,C\} = B$;
(3) $Q := \max\{A,B,C\} = A$ and $T := \min\{A,B,C\} = C$.

For case (1), we need $B \geq A$, and thus we must have

$$\alpha + b + \beta + \gamma + \delta - 3 \geq a + \alpha + b + \gamma + \delta - 4 \Longrightarrow \beta \geq a - 1.$$

Note that if $\beta = a - 1$ and $B \geq A$, then $A = B$; we will address this situation in case (3) where $A \geq B$. Thus, we will assume, without loss of generality, that $\beta \geq a$ which gives us $b \geq \beta \geq a \geq \alpha$. Furthermore, we note that $b \neq \alpha$ ($b > \alpha$) since, if $b = \alpha$ then

$$b \geq \beta \geq a \geq \alpha \Longrightarrow b = \beta = a = \alpha \Rightarrow R/I \text{ is a type 1 algebra.}$$

Using Lemma 2.1, we have

$$\begin{aligned}\Delta H_{R/I} &= H(a-\alpha,\gamma,\delta)(-\alpha) + H(\alpha, b-\beta,\gamma)(-\beta)\\ &\quad + H(\alpha,c,d) - H(a-\alpha,\gamma,\delta)(-\alpha-b)\\ &\quad - H(\alpha,b-\beta,\gamma)(-\beta-\delta) - H(\alpha,c,d)(-\beta)\\ &=: P_1 + P_2 + P_3 - N_1 - N_2 - N_3.\end{aligned}$$

Since each of the P_i and N_i are the Hilbert functions of complete intersections, we know that they are strictly unimodal and peak in the middle degree. Ideally, we want to show that the middle (peak) degree of each of the P_i is less than C, and the middle degree of each of the N_i is greater than B. If so, then segments ΔP_i and $-\Delta N_i$ will be zero or negative between C and B, implying that $\Delta^2 H_{R/I} \leq 0$ between these degrees. Note that

$$\begin{aligned}
&a+\alpha+\gamma+\delta-3 \leq a+b+\gamma+\delta-4 = \\
&\qquad \alpha+\beta+c+d-4 \text{ (since } \alpha < b) \Longrightarrow && P_1 \text{ peaks before } C\\
&\alpha+c+d-3 \leq \alpha+\beta+c+d-4 \Longrightarrow && P_3 \text{ peaks before } C\\
&a+\alpha+\gamma+\delta+2b-3 \geq \alpha+b+\beta+\gamma+\delta-4 \Longrightarrow && N_1 \text{ peaks after } B\\
&\alpha+b+\beta+\gamma+2\delta-3 \geq \alpha+b+\beta+\gamma+\delta-4 \Longrightarrow && N_2 \text{ peaks after } B
\end{aligned}$$

The above inequalities give that P_1, P_3, N_1 and N_2 are non-positive between C and B. This leaves $\Delta[P_2 - N_3]$ to check. We note that, since P_2 and N_3 start in the same degree and $c \geq \gamma$ and $d \geq b$ (so $d > b-\beta$), the difference of these two Hilbert functions will initially be zero and then decrease until at least the peak value of N_3. We notice that this peak value is at

$$\left\lfloor \frac{\alpha+c+d+2\beta-3}{2} \right\rfloor,$$

which is greater than or equal to B (since $\alpha+c+d+2\beta-3 = a+b+\gamma+\delta+\beta-3 \geq \alpha+b+\beta+\gamma+\delta-4$). Thus, $\Delta[P_2 - N_3]$ is non-positive until at least degree B. Thus, we have that $\Delta^2 H_{R/I}$ is zero or negative between A and B. This implies that $\Delta H_{R/I}$ is positive, possibly zero, then negative so $H_{R/I}$ is strictly unimodal.

For cases (2) and (3), without loss of generality, we have the following assumptions (in addition to our previous assumption that $d \geq c, d \geq b$

and $d \geq a$):

(a) $a > \beta$ and $b > \alpha$ (if either inequality fails, change the variables so that $\beta \geq a$ to be in case (1).
(b) $b \geq a$ (if $b < a$ then swap the variables x and y).

For case (2), $A \geq C \geq B$ so we want to show that $\Delta^2 H_{R/I} \leq 0$ between B and A. To do this, we will change the decomposition of the first difference of the Hilbert function of R/I using Lemma 2.1. In particular, we will change P_3 and N_3. We note that this does not change the values of A, B or C. Our new decomposition will be

$$\begin{aligned}\Delta H_{R/I} &= H(a-\alpha,\gamma,\delta)(-\alpha) + H(\alpha,\gamma,\delta)(-\beta)\\ &\quad + H(\alpha,\beta,c) - H(a-\alpha,\gamma,\delta)(-\alpha-b)\\ &\quad - H(\alpha,\gamma,\delta)(-b) - H(\alpha,\beta,c)(-d)\\ &=: P_1 + P_2 + P_3 - N_1 - N_2 - N_3.\end{aligned}$$

With the above assumptions and decomposition, we have that the P_i peak before B and the N_i peak after A. Indeed,

$$\begin{aligned}
&a+\alpha+\gamma+\delta-3 \leq \alpha+b+\beta+\gamma+\delta-4\\
&\quad (b \geq a) \Longrightarrow && P_1 \text{ peaks before } B\\
&\alpha+\gamma+\delta+2\beta-3 \leq \alpha+b+\beta+\gamma+\delta-4\\
&\quad (b \geq a, a > \beta \Longrightarrow b > \beta) \Longrightarrow && P_2 \text{ peaks before } B\\
&\alpha+\beta+c-3 \leq 2\alpha+2\beta+c+d-a-4 =\\
&\quad \alpha+b+\beta+\gamma+\delta-4(d \geq a) \Longrightarrow && P_3 \text{ peaks before } B\\
&a+\alpha+\gamma+\delta+2b-3 \geq a+\alpha+b+\gamma+\delta-4 \Longrightarrow && N_1 \text{ peaks after } A\\
&\alpha+\gamma+\delta+2b-3 \geq a+\alpha+b+\gamma+\delta-4\\
&\quad (b \geq a)) \Longrightarrow && N_2 \text{ peaks after } A\\
&\alpha+\beta+c+2d-3 \geq 2\alpha+\beta+c+d-4 =\\
&\quad a+\alpha+b+\gamma+\delta-4\,(d \geq a) \Longrightarrow && N_3 \text{ peaks after } A
\end{aligned}$$

Since ΔP_i and $-\Delta N_i$ are zero or negative between B and A for all degrees i, $\Delta^2 H_{R/I} \leq 0$ between these degrees. Thus, $\Delta H_{R/I}$ is decreasing or constant between B and A. This implies that $H_{R/I}$ is strictly unimodal, which completes case (2).

For case (3), $A \geq B \geq C$ so we want to show that $\Delta^2 H_{R/I} \leq 0$ between degrees C and A. We note that, since $B \geq C$, we have

$$\begin{aligned} \alpha + b + \beta + \gamma + \delta - 3 &\geq \alpha + \beta + c + d - 4 \\ &= a + b + \gamma + \delta - 4 \\ &\Longrightarrow \alpha + \beta \geq a - 1. \end{aligned}$$

We note that, if $\alpha + \beta = a - 1$ and $B \geq C$ then $B = C$. Since the case $A > B = C$ has already been addressed in case (2) above, we will assume without loss of generality that

$$\alpha + \beta \geq a.$$

To address case (3), we will change the decomposition of the first difference of the Hilbert function to be

$$\begin{aligned} \Delta H_{R/I} &= H(a-\alpha, \gamma, \delta)(-\alpha) + H(b-\beta, \gamma, \delta)(-\beta) \\ &\quad + H(\alpha, \beta, c) - H(a-\alpha, \gamma, \delta)(-\alpha - b) \\ &\quad - H(b-\beta, \gamma, \delta)(-\alpha-\beta) - H(\alpha, \beta, c)(-d) \\ &=: P_1 + P_2 + P_3 - N_1 - N_2 - N_3. \end{aligned}$$

With these assumptions and decomposition, we note that the P_i peak before C and the N_i peak after A. Indeed,

$$\begin{aligned} &a + \alpha + \gamma + \delta - 3 \leq a + b + \gamma + \delta - 4 = \\ &\quad \alpha + \beta + c + d - 4 \ (b > \alpha) \Longrightarrow && P_1 \text{ peaks before } C \\ &b + \beta + \gamma + \delta - 3 \leq a + b + \gamma + \delta - 4 = \\ &\quad \alpha + \beta + c + d - 4 \ (a > \beta) \Longrightarrow && P_2 \text{ peaks before } C \\ &\alpha + \beta + c - 3 \leq \alpha + \beta + c + d - 4 \Longrightarrow && P_3 \text{ peaks before } C \\ &a + \alpha + \gamma + \delta + 2b - 3 \geq a + \alpha + b + \gamma + \delta - 4 \Longrightarrow && N_1 \text{ peaks after } A \\ &b + \beta + \gamma + \delta + 2\alpha - 3 \geq a + \alpha + b + \gamma + \delta - 4 \\ &\quad (\beta + \alpha \geq a) \Longrightarrow && N_2 \text{ peaks after } A \\ &\alpha + \beta + c + 2d - 3 \geq 2\alpha + \beta + c + d - 4 = \\ &\quad a + \alpha + b + \gamma + \delta - 4 \ (d \geq \alpha) \Longrightarrow && N_3 \text{ peaks after } A. \end{aligned}$$

Since ΔP_i and $-\Delta N_i$ are zero or negative between C and A for all degrees i, $\Delta^2 H_{R/I} \leq 0$ between these degrees, so $\Delta H_{R/I}$ is decreasing or constant between C and A. Thus, we have that $H_{R/I}$ is strictly unimodal, which completes the proof for case (3) and Proposition 3.4. □

In addition to the unimodality of the Hilbert function of Artinian level monomial algebras, we have found interesting results related to the licciness of these algebras.

Remark 3.5. Let $a \geq \alpha$, $b \geq \beta$, $c \geq \gamma$ and $d \geq \delta$.

(i) The ideal $(x^a, y^b, z^c, w^d, x^\alpha w^\delta, y^\beta w^\delta, z^\gamma w^\delta)$ with $a+b+c+\delta = \alpha+\beta+\gamma+d, d > \delta$ is licci.

(ii) Assume that $a \neq \alpha$, $b \neq \beta$, $c \neq \gamma$ and $d \neq \delta$; if any of these equalities hold, then this ideal falls into case (1) above. The ideal $(x^a, y^b, z^c, w^d, x^\alpha z^\gamma, x^\alpha w^\delta, y^\beta z^\gamma, y^\beta w^\delta)$ where $a+b+\gamma+\delta = \alpha+\beta+c+d$ is not licci.

Proof. For ideal (i), we will show that the ideal is licci by constructing the CI-links. The ideal decomposes as

$$I = (x^a, y^b, z^c, w^d) + w^\delta(x^\alpha, y^\beta, z^\gamma, w^{d-\delta}).$$

Lemma 2.5 from **[10]** gives that I is CI-linked to $T := (x^\alpha, y^\beta, z^\gamma, w^{d-\delta})$ by the double link defined by the monomial regular sequences $C = (x^a, y^b, z^c, w^d)$ and $S = (x^a, y^b, z^c, w^{d-\gamma})$. Thus, $I \overset{C}{\sim} Y \overset{S}{\sim} T$ where Y is some monomial ideal in R, so I is licci.

Ideal (ii) decomposes as

$$I = (x^a, y^b, z^c, w^d) + (x^\alpha z^\gamma, x^\alpha w^\delta, y^\beta z^\gamma, y^\beta w^\delta).$$

Since the second piece of this ideal is an ideal of height at least two, Lemma 2.4 from **[10]** gives us that the original ideal cannot be licci. However, we conjecture that ideal (ii) is glicci. □

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