1-15-2002

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ELSEVIER

Linear Algebra and its Applications 341 (2002) 45–56

LINEAR ALGEBRA AND ITS APPLICATIONS

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On graphs with equal algebraic and vertex connectivity

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Received 10 August 2000; accepted 31 December 2000

Submitted by C.-K. Li

Dedicated to T. Ando

Abstract

Let $\mathscr{G}$ be an undirected unweighted graph on n vertices, let L be its Laplacian matrix, and let $L^{\#} = (\ell^{\#}_{i,j})$ be the group inverse of L. It is known that for $\mathscr{Z}(L^{\#}) := (1/2)\max_{1\leqslant i,j\leqslant n}\sum_{s=1}^{n}|\ell^{\#}_{i,s} - \ell^{\#}_{j,s}|$, the quantity $1/\mathscr{Z}(L^{\#})$ is a lower bound on the algebraic connectivity $a(\mathscr{G})$ of $\mathscr{G}$, while the vertex connectivity of $\mathscr{G}$, $v(\mathscr{G})$, is an upper bound on $a(\mathscr{G})$. We characterize the graphs $\mathscr{G}$ for which $v(\mathscr{G}) = a(\mathscr{G})$ and subsequently prove that if $n \geqslant v(\mathscr{G})^2$, then $v(\mathscr{G}) = a(\mathscr{G})$ holds if and only if $1/\mathscr{Z}(L^{\#}) = a(\mathscr{G}) = v(\mathscr{G})$. We close with an example showing that the equality $1/\mathscr{Z}(L^{\#}) = a(\mathscr{G})$ does not necessarily imply that $1/\mathscr{Z}(L^{\#}) = a(\mathscr{G}) = v(\mathscr{G})$. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 5C50; 15A48

Keywords: Undirected graph; Algebraic connectivity; Vertex connectivity; Laplacian matrix

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[1] Research supported by NSERC Grant No. OGP0138251.

[2] The work of this author was supported in part by NSF Grant No. DMS9973247.

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1. Introduction

An *undirected graph* $\mathscr{G} = (V, E)$ on n vertices is a finite set V of cardinality n, whose elements are called *vertices*, together with a set E of two-element subsets of V called *edges*. It will be convenient to label the vertices by $1, \ldots, n$. Associated with $\mathscr{G}$ is its *Laplacian matrix* $L = (\ell_{i,j})$ which is defined as follows:

$$\ell_{i,j} = \begin{cases} -1 & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ \;\;0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ -\sum_{k \neq i} \ell_{i,k} & \text{if } i = j. \end{cases}$$

It is known that the Laplacian matrix is a symmetric positive semidefinite M-matrix, and we take its eigenvalues to be arranged in nondescending order: $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. Fiedler [4, p. 298] showed that $\lambda_2 > 0$ if and only if $\mathscr{G}$ is connected, and as a result, $a(\mathscr{G}) := \lambda_2$ is known as the *algebraic connectivity* of $\mathscr{G}$. In that paper Fiedler also considers the *vertex connectivity* of $\mathscr{G}$, $v(\mathscr{G})$—the minimal number of vertices whose removal yields a disconnected graph—and shows that if $\mathscr{G} \neq K_n$ (where K_n is the complete graph on n vertices), then

$$a(\mathscr{G}) \leqslant v(\mathscr{G}). \tag{1.1}$$

Another inequality involving $a(\mathscr{G})$ is studied in [8,9]. These papers rely on [7] which discusses how $a(G)$ can be studied both algebraically and graphically via the group inverse $L^{\#}(\mathscr{G})$.[3] For a matrix $B \in \mathbb{C}^{n,n}$, define the quantity $\mathscr{Z}(B)$ by

$$\mathscr{Z}(B) := \frac{1}{2} \max_{1 \leqslant i,j \leqslant n} \sum_{s=1}^{n} |b_{i,s} - b_{j,s}| = \frac{1}{2} \max_{1 \leqslant i,j \leqslant n} \left\| e_i^{\mathrm{T}} B - e_j^{\mathrm{T}} B \right\|_1, \tag{1.2}$$

where e_k, $k = 1, \ldots, n$, denote the usual unit coordinate vectors in $\mathbb{C}^n$. If $\mathscr{G}$ is a connected graph with Laplacian matrix L and if $L^{\#}$ is its group inverse, then we have the following lower bound on $a(\mathscr{G})$:

$$\frac{1}{\mathscr{Z}(L^{\#}(\mathscr{G}))} \leqslant a(\mathscr{G}). \tag{1.3}$$

We comment that (1.3) is a consequence of a more general theorem, due to Bauer et al. [1] (but see also [3,13], and [14, p. 63]), providing an upper bound on the modulus of eigenvalues of a matrix with constant row sums. We note that these references deal with the more general bound in the context of Markov processes. There, if $B \in \mathbb{R}^{n,n}$ is a transition matrix for an regular Markov chain, then (1.2) is known as the *coefficient of ergodicity of the chain*. We further mention that, for such a transition matrix B, the group inverse of $I - B$ plays a central role in the computation of various parameters important for the chain. As references for this we give here [2,11,12] and and some of the references cited therein.

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[3] Since $L^{\#}(\mathscr{G})$ is symmetric, $L^{\#}(\mathscr{G})$ coincides with the Moore–Penrose inverse of $L(\mathscr{G})$. We choose to use the terminology of group inverse, as many of the results we use arose in the context of group inverses.

In [8,9] some properties of the bound (1.3) are developed, while Ref. [6] shows that equality holds in (1.3) for the class of so-called *maximal graphs* (see [10] for more on maximal graphs). The goal of this paper is to better understand the inequalities (1.1) and (1.3). Specifically, we characterize the graphs $\mathscr{G}$ such that

$$a(\mathscr{G}) = v(\mathscr{G}). \tag{1.4}$$

We then determine conditions on $\mathscr{G}$ under which

$$a(\mathscr{G}) = v(\mathscr{G}) \iff \frac{1}{\mathscr{Z}\left(L^{\#}(\mathscr{G})\right)} = a(\mathscr{G}) = v(\mathscr{G}). \tag{1.5}$$

Finally, we show through an example that in general,

$$\frac{1}{\mathscr{Z}\left(L^{\#}(\mathscr{G})\right)} = a(\mathscr{G}) \;\not\Rightarrow\; \frac{1}{\mathscr{Z}\left(L^{\#}(\mathscr{G})\right)} = a(\mathscr{G}) = v(\mathscr{G}). \tag{1.6}$$

2. Main results

We begin with some terminology and notation. If $\mathscr{G}_1 = (V_1, E_1)$ and $\mathscr{G}_2 = (V_2, E_2)$ are two graphs on disjoint sets of vertices, their *union*, $\mathscr{G}_1 + \mathscr{G}_2$, is the graph $(V_1 \cup V_2, E_1 \cup E_2)$. The *join*, $\mathscr{G}_1 \vee \mathscr{G}_2$, of $\mathscr{G}_1$ and $\mathscr{G}_2$ is the graph obtained from $\mathscr{G}_1 + \mathscr{G}_2$ by adding new edges from each vertex in $\mathscr{G}_1$ to every vertex of $\mathscr{G}_2$. Throughout the paper we use the bold face $\mathbf{1}$ to denote the vector of all ones of the appropriate size, but occasionally, for the sake of clarity, we will subindex $\mathbf{1}$ as well as the identity matrix and the matrix J of all ones by the integer indicating their size.

We begin with a theorem characterizing the graphs $\mathscr{G}$ for which equality holds in (1.1).

Theorem 2.1. *Let $\mathscr{G}$ be a non-complete, connected graph on n vertices. Then $v(\mathscr{G}) = a(\mathscr{G})$ if and only if $\mathscr{G}$ can be written as $\mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ is a disconnected graph on $n - v(\mathscr{G})$ vertices and $\mathscr{G}_2$ is a graph on $v(\mathscr{G})$ vertices with $a(\mathscr{G}_2) \geqslant 2v(\mathscr{G}) - n$.*

Proof. First suppose that $v(\mathscr{G}) = a(\mathscr{G})$. Then by simultaneously permuting rows and columns, the Laplacian matrix of $\mathscr{G}$ can be written as

$$L(\mathscr{G}) = \left[\begin{array}{c|c|c} L(\mathscr{H}_1) + D_1 & 0 & -X \\ \hline 0 & L(\mathscr{H}_2) + D_2 & -Y \\ \hline -X^{\mathrm{T}} & -Y^{\mathrm{T}} & L(\mathscr{G}_2) + D_3 \end{array}\right],$$

where the (1, 1)-, (1, 2)-, (2, 1)-, and (2, 2)-blocks, together, form an $(n - v(\mathscr{G})) \times (n - v(\mathscr{G}))$ matrix; the (3, 3)-block is a $v(\mathscr{G}) \times v(\mathscr{G})$ matrix; and where $\mathscr{H}_1 + \mathscr{H}_2$ is the graph (necessarily disconnected) on $n - v(\mathscr{G})$ vertices formed from $\mathscr{G}$ by deleting a collection of $v(\mathscr{G})$ suitable vertices; $L(\mathscr{H}_1)$ and $L(\mathscr{H}_2)$ are the corresponding Laplacian matrices, and D_1 and D_2 are suitable diagonal matrices. Suppose that

$\mathscr{H}_1$ has n_1 vertices and that $\mathscr{H}_2$ has n_2 vertices, where $n_1 + n_2 = n - v(\mathscr{G})$. Let $w^{\mathrm{T}} = [\ n_2\mathbf{1}_{n_1}^{\mathrm{T}} \mid -n_1\mathbf{1}_{n_2}^{\mathrm{T}} \mid 0^{\mathrm{T}}\ ]$ and note that $w^{\mathrm{T}}\mathbf{1} = 0$. Now

$$\begin{aligned} w^{\mathrm{T}}L(\mathscr{G})w &= n_2^2\mathbf{1}_{n_1}^{\mathrm{T}}(L(\mathscr{H}_1) + D_1)\mathbf{1}_{n_1} + n_1^2\mathbf{1}_{n_2}^{\mathrm{T}}(L(\mathscr{H}_2) + D_2)\mathbf{1}_{n_2} \\ &= n_2^2\mathbf{1}^{\mathrm{T}}D_1\mathbf{1} + n_1^2\mathbf{1}^{\mathrm{T}}D_2\mathbf{1}. \end{aligned} \tag{2.1}$$

Notice that each of D_1 and D_2 is a diagonal matrix with diagonal entries at most $v(\mathscr{G})$. Thus we have that

$$w^{\mathrm{T}}L(\mathscr{G})w \leqslant n_2^2 n_1 v(\mathscr{G}) + n_1^2 n_2 v(\mathscr{G}) = v(\mathscr{G})w^{\mathrm{T}}w, \tag{2.2}$$

with equality if and only if $D_1 = v(\mathscr{G})I_{n_1}$ and $D_2 = v(\mathscr{G})I_{n_2}$. Now, since $w^{\mathrm{T}}\mathbf{1} = 0$, it follows from the Courant–Fischer minimax principle (see [5, p. 179]) that $a(\mathscr{G})w^{\mathrm{T}}w \leqslant w^{\mathrm{T}}L(\mathscr{G})w$. Hence, together with (2.2) we can write that

$$v(\mathscr{G})w^{\mathrm{T}}w = a(\mathscr{G})w^{\mathrm{T}}w \leqslant w^{\mathrm{T}}L(\mathscr{G})w \leqslant v(\mathscr{G})w^{\mathrm{T}}w$$

and so we must have that $D_1 = v(\mathscr{G})I_{n_1}$ and $D_2 = v(\mathscr{G})I_{n_2}$. Thus $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1 = \mathscr{H}_1 + \mathscr{H}_2$ is a disconnected graph. As a result,

$$L(\mathscr{G}) = \left[\begin{array}{c|c} L(\mathscr{G}_1) + v(\mathscr{G})I & -J \\ \hline -J & L(\mathscr{G}_2) + (n - v(\mathscr{G}))I \end{array}\right]. \tag{2.3}$$

Suppose that λ is an eigenvalue of $L(\mathscr{G}_1)$ having an eigenvector x such that $x^{\mathrm{T}}\mathbf{1} = 0$ (note that $L(\mathscr{G}_1)$ has $n - v(\mathscr{G}) - 1$ such linearly independent eigenvectors). Then $[\ x^{\mathrm{T}} \mid 0^{\mathrm{T}}\ ]^{\mathrm{T}}$ is an eigenvector for $L(\mathscr{G})$ corresponding to $\lambda + v(\mathscr{G})$. It follows that $n - 2$ of the eigenvalues of $L(\mathscr{G})$ are of the form $\lambda + v(\mathscr{G})$, where $\lambda \in \sigma(L(\mathscr{G}_1))$ with an eigenvector orthogonal to $\mathbf{1}$, or $\gamma + n - v(\mathscr{G})$, where $\gamma \in \sigma(L(\mathscr{G}_2))$ with an eigenvector orthogonal to $\mathbf{1}$. Observe that 0 is also an eigenvalue of $L(\mathscr{G})$, as is n, the latter with eigenvector

$$\left[v(\mathscr{G})\mathbf{1}_{n-v(\mathscr{G})}^{\mathrm{T}} \,\middle|\, -(n - v(\mathscr{G}))\mathbf{1}_{v(\mathscr{G})}^{\mathrm{T}}\right]^{\mathrm{T}}.$$

Consequently,

$$\begin{aligned} a(\mathscr{G}) &= \min\left\{v(\mathscr{G}) + a(\mathscr{G}_1),\ n - v(\mathscr{G}) + a(\mathscr{G}_2)\right\} \\ &= \min\left\{v(\mathscr{G}),\ n - v(\mathscr{G}) + a(\mathscr{G}_2)\right\} \end{aligned}$$

(since $\mathscr{G}_1$ is disconnected). We thus conclude that $a(\mathscr{G}_2) \geqslant 2v(\mathscr{G}) - n$, since $a(\mathscr{G}) = v(\mathscr{G})$.

Conversely, suppose that $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ is a disconnected graph on $n - v(\mathscr{G})$ vertices and $\mathscr{G}_2$ is a graph on $v(\mathscr{G})$ vertices with $a(\mathscr{G}_2) \geqslant 2v(\mathscr{G}) - n$. Then $L(\mathscr{G})$ has the form as in (2.3). Therefore, by the preceding analysis, we see that

$$\begin{aligned} a(\mathscr{G}) &= \min\left\{\{v(\mathscr{G}) + a(\mathscr{G}_1),\ n - v(\mathscr{G}) + a(\mathscr{G}_2)\right\} \\ &= \min\left\{v(\mathscr{G}),\ n - v(\mathscr{G}) + a(\mathscr{G}_2)\right\} \end{aligned}$$

(since $\mathscr{G}_1$ is disconnected). Since $a(\mathscr{G}_2) \geqslant 2v(\mathscr{G}) - n$, it follows that $a(\mathscr{G}) = v(\mathscr{G})$. □

We now briefly discuss the conditions given in Theorem 2.1. The condition that $\mathscr{G}$ can be written as $\mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ is a disconnected graph on $n - v(\mathscr{G})$ vertices, is a purely graph theoretic condition. The following shows that it is simple to decide whether $\mathscr{G}$ satisfies the graph theoretic condition, and if so to determine $v(\mathscr{G})$, $\mathscr{G}_1$ and $\mathscr{G}_2$. Consider $\overline{\mathscr{G}}$, the graph which is the complement of $\mathscr{G}$. If $\overline{\mathscr{G}}$ is connected, then $\mathscr{G}$ is not a join of graphs. Otherwise, let $\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_t$ be the connected components of $\overline{\mathscr{G}}$. It can be verified that the only possibility for $\mathscr{G}_1$ is an $\overline{\mathscr{H}_i}$. Thus, if each $\overline{\mathscr{H}_i}$ is connected, then $\mathscr{G}$ does not satisfy the graph theoretic condition. Otherwise, choose $\mathscr{H}_j$ so that $\mathscr{H}_j$ has the largest number of vertices among the $\mathscr{H}_i$ for which $\overline{\mathscr{H}_i}$ is disconnected. Then the graph theoretic condition is satisfied with $\mathscr{G}_1 = \overline{\mathscr{H}_j}$ and $\mathscr{G}_2$ the induced subgraph on the vertices of $\mathscr{G}$ not in $\mathscr{G}_1$.

Now assume that $\mathscr{G}$ satisfies the graph theoretic condition; that is, assume that $\mathscr{G}$ can be written as $\mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ is a disconnected graph on $n - v(\mathscr{G})$ vertices. The second condition, $a(\mathscr{G}_2) \geqslant 2v(\mathscr{G}) - n$, is a spectral condition. If $v(\mathscr{G}) \leqslant n/2$, then we see that the spectral condition holds without having to compute $a(\mathscr{G}_2)$. However, if $v(\mathscr{G}) > n/2$, then we must either compute $a(\mathscr{G}_2)$ or use known bounds on $a(\mathscr{G}_2)$ to compare $a(\mathscr{G}_2)$ and $2v(\mathscr{G}) - n$. For example, if the minimum degree of a vertex in $\mathscr{G}_2$ is less than $2v(\mathscr{G}) - n$, then the spectral condition is not satisfied. Note that for fixed $v(\mathscr{G})$ the spectral condition weakens as a n increases. Thus, if $\mathscr{G}_2$ is has v vertices, then $v(\mathscr{G}_1 \vee \mathscr{G}_2) = a(\mathscr{G}_1 \vee \mathscr{G}_2)$ for each disconnected graph $\mathscr{G}_1$ with at least $v - a(\mathscr{G}_2)$ vertices.

Theorem 2.1 gives necessary and sufficient conditions for a graph to satisfy $v(\mathscr{G}) = a(\mathscr{G})$. Next we determine the graphs $\mathscr{G}$ for which

$$v(\mathscr{G}) = \frac{1}{\mathscr{Z}(L^{\#}(\mathscr{G}))}.$$

Since having

$$v(\mathscr{G}) = \frac{1}{\mathscr{Z}(L^{\#}(\mathscr{G}))}$$

requires that $v(\mathscr{G}) = a(\mathscr{G})$, Theorem 2.1 is applicable and this is reflected in the conditions of the following lemma.

Lemma 2.2. *Suppose that $\mathscr{G}$ is a graph on n vertices such that $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ and $\mathscr{G}_2$ are graphs on $n - v(\mathscr{G})$ vertices and $v(\mathscr{G})$ vertices, respectively, and where $\mathscr{G}_1$ is a disconnected graph. Then*

$$L^{\#}(\mathscr{G}) = \left[\begin{array}{c|c} [L(\mathscr{G}_1) + v(\mathscr{G})I]^{-1} - \frac{n+v(\mathscr{G})}{v(\mathscr{G})n^2} J & -\frac{1}{n^2} J \\ \hline -\frac{1}{n^2} J & [L(\mathscr{G}_2) + (n - v(\mathscr{G}))I]^{-1} - \frac{2n-v(\mathscr{G})}{(n-v(\mathscr{G}))n^2} J \end{array}\right],$$

where the $(1, 1)$-block is an $(n - v(\mathscr{G})) \times (n - v(\mathscr{G}))$ matrix and the $(2, 2)$-block is a $v(\mathscr{G}) \times v(\mathscr{G})$ matrix.

Proof. As $L(\mathscr{G}_1)$ and $L(\mathscr{G}_2)$ are both singular M-matrices, $L(\mathscr{G}_1)+v(\mathscr{G})I$ and $L(\mathscr{G}_2)+(n-v(\mathscr{G}))I$ are both nonsingular M-matrices and hence they are invertible and their inverses are nonnegative matrices. We have that

$$L(\mathscr{G})=\left[\begin{array}{c|c} L(\mathscr{G}_1)+v(\mathscr{G})I & -J \\ \hline -J & L(\mathscr{G}_2)+(n-v(\mathscr{G}))I \end{array}\right],$$

where the (1, 1)-block is an $(n-v(\mathscr{G}))\times(n-v(\mathscr{G}))$ matrix and the (2, 2)-block is a $v(\mathscr{G})\times v(\mathscr{G})$ matrix. Letting M be the right-hand side above, we find that $L(\mathscr{G})M = ML(\mathscr{G}) = I-\frac{1}{n}J$ and $MJ=JM=0$. Thus M satisfies the defining properties (see [3]) of the group inverse of L, namely, $MLM=M$, $LML=L$, and both ML and LM are Hermitian. $\square$

Since $\mathscr{Z}(A+bJ)=\mathscr{Z}(A)$ for any square matrix A and scalar b, Lemma 2.2 immediately yields the following corollary.

Corollary 2.3. *Let $\mathscr{G}$ be a graph on n vertices such that $\mathscr{G}=\mathscr{G}_1\vee\mathscr{G}_2$, where $\mathscr{G}_1$ and $\mathscr{G}_2$ are graphs on $n-v(\mathscr{G})$ vertices and $v(\mathscr{G})$ vertices, respectively, and where $\mathscr{G}_1$ is a disconnected graph. Then*

$$\begin{aligned}
&\mathscr{Z}(L^{\#}(\mathscr{G}))\\
&\quad=\max\Bigg\{\mathscr{Z}\left([L(\mathscr{G}_1)+v(\mathscr{G})I]^{-1}\right),\ \mathscr{Z}\left([L(\mathscr{G}_2)+(n-v(\mathscr{G}))I]^{-1}\right),\\
&\qquad\max_{\substack{1\leqslant i\leqslant n-v(\mathscr{G})\\ n-v(\mathscr{G})+1\leqslant j\leqslant n}}\left\{\frac{1}{2}\left[\left\|e_i^{\mathrm{T}}((L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}-\frac{1}{nv(\mathscr{G})}J)\right\|_1\right.\right.\\
&\qquad\qquad+\left\|e_j^{\mathrm{T}}((L(\mathscr{G}_2)+(n-v(\mathscr{G}))I)^{-1}\right.\\
&\qquad\qquad\qquad\left.\left.\left.-\frac{1}{(n-v(\mathscr{G}))n}J)\right\|_1\right]\right\}\Bigg\}.
\end{aligned}\tag{2.4}$$

In order to establish, under the conditions of Corollary 2.3, which of the three expressions in the right-hand side of (2.4) yields the maximum, we need to obtain more precise information about the matrices $[L(\mathscr{G}_1)+v(\mathscr{G})I]^{-1}$ and $[L(\mathscr{G}_2)+(n-v(\mathscr{G}))I]^{-1}$. Observe that both of these matrices are inverses of matrices in which the diagonal entries of a Laplacian matrix are perturbed. The following lemma and subsequent corollary give us useful information about the diagonals of the inverses of these matrices.

Lemma 2.4. *Let $\mathscr{H}$ be a graph and $m>0$. Form $\mathscr{H}'$ from $\mathscr{H}$ by adding an edge. Then*

$$\operatorname{diag}\left(\left[L(\mathscr{H}')+mI\right]^{-1}\right)\leqslant\operatorname{diag}\left([L(\mathscr{H})+mI]^{-1}\right).$$

Proof. $L(\mathscr{H}') = L(\mathscr{H}) + xx^{\mathrm{T}}$ for some vector x so that

$$\begin{aligned}&\left[L(\mathscr{H}') + mI\right]^{-1}\\ &\quad= \left[L(\mathscr{H}) + mI + xx^{\mathrm{T}}\right]^{-1}\\ &\quad= [L(\mathscr{H}) + mI]^{-1} - \frac{(L(\mathscr{H}) + mI)^{-1}xx^{\mathrm{T}}(L(\mathscr{H}) + mI)^{-1}}{1 + x^{\mathrm{T}}(L(\mathscr{H}) + mI)^{-1}x}.\end{aligned}$$

Since $[L(\mathscr{H}) + mI]^{-1}$ is positive definite, the result now follows by evaluating $e_i^{\mathrm{T}}[L(\mathscr{H}') + mI]^{-1}e_i$ for each i. □

Corollary 2.5. *Let $\mathscr{G}$ be a graph on n vertices such that $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ and $\mathscr{G}_2$ are graphs on $n - m$ vertices and m vertices, respectively, and where $m > 0$. Then*

$$\operatorname{diag}\Big([L(\mathscr{G}_1) + mI]^{-1}\Big) > \frac{m+1}{nm}I.$$

Proof. By Lemma 2.4,

$$\begin{aligned}\operatorname{diag}\Big([L(\mathscr{G}_1) + mI]^{-1}\Big) &\geqslant \operatorname{diag}\Big(\left[L(K_{n-m}) + mI\right]^{-1}\Big)\\ &= \operatorname{diag}\Big([nI - J]^{-1}\Big) > \operatorname{diag}\Big([(n+m)I - J]^{-1}\Big)\\ &= \operatorname{diag}\left(\frac{1}{n}\left[I + \frac{1}{m}J\right]\right)\\ &= \frac{m+1}{nm}I. \qquad \square\end{aligned}$$

Lemma 2.4 and Corollary 2.5 gave us necessary information about the first two quantities in the braces on the right-hand side of (2.4). We now consider the third quantity there.

Lemma 2.6. *Let $\mathscr{G}$ be a graph on n vertices such that $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ and $\mathscr{G}_2$ are graphs on $n - v(\mathscr{G})$ vertices and $v(\mathscr{G})$ vertices, respectively, and where $\mathscr{G}_1$ is a disconnected graph. Then*

$$\begin{aligned}\frac{1}{n} &\leqslant \max_{1\leqslant i\leqslant n-v(\mathscr{G})}\left\|e_i^{\mathrm{T}}\left([L(\mathscr{G}_1) + v(\mathscr{G})I]^{-1} - \frac{1}{nv(\mathscr{G})}J\right)\right\|_1\\ &\leqslant \frac{n - v(\mathscr{G}) - 2}{nv(\mathscr{G})} + \frac{1}{v(\mathscr{G})}\end{aligned} \tag{2.5}$$

and

$$\frac{1}{n} \leqslant \max_{1\leqslant j\leqslant v(\mathscr{G})}\left\|e_j^{\mathrm{T}}\left([L(\mathscr{G}_2) + (n - v(\mathscr{G}))I]^{-1} - \frac{1}{n(n - v(\mathscr{G}))}J\right)\right\|_1$$

$$\leqslant \frac{v(\mathscr{G})-2}{nv(\mathscr{G})} + \frac{1}{(n-v(\mathscr{G}))}. \tag{2.6}$$

Proof. For the lower bound in (2.5), note that

$$[L(\mathscr{G}_1)+v(\mathscr{G})I]^{-1} - \frac{1}{nv(\mathscr{G})}J = (L(\mathscr{G}_1)+v(\mathscr{G})I+J)^{-1}.$$

Hence

$$\max_{1\leqslant i\leqslant n-v(\mathscr{G})} \left\| e_i^{\mathrm{T}} \left((L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1} - \frac{1}{nv(\mathscr{G})}J \right) \right\|_1$$
$$\geqslant \max_{1\leqslant i\leqslant n-v(\mathscr{G})} \left\{ e_i^{\mathrm{T}} \left([(L(\mathscr{G}_1)+v(\mathscr{G})I]^{-1} - \frac{1}{nv(\mathscr{G})}J \right) \mathbf{1} \right\}$$
$$= \max_{1\leqslant i\leqslant n-v(\mathscr{G})} \left\{ e_i^{\mathrm{T}} [L(\mathscr{G}_1)+v(\mathscr{G})I+J]^{-1} \mathbf{1} \right\} = \frac{1}{n}.$$

Next, we consider the upper bound in (2.5). According to Corollary 2.5, the ith diagonal entry of $(L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}$ is at least

$$\frac{v(\mathscr{G})+1}{nv(\mathscr{G})} > \frac{1}{nv(\mathscr{G})}.$$

Also, we have that $e_i^{\mathrm{T}}(L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}\mathbf{1} = (1/v(\mathscr{G}))$. Now write $e_i^{\mathrm{T}}(L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}$ as $de_i^{\mathrm{T}}+x^{\mathrm{T}}$, where x^{T} has a 0 in its ith position and where $x^{\mathrm{T}} \geqslant 0$. Then we have that

$$\left\| e_i^{\mathrm{T}} [(L(\mathscr{G}_1)+v(\mathscr{G})I]^{-1} - \frac{1}{nv(\mathscr{G})}J) \right\|_1$$
$$= \left\| de_i^{\mathrm{T}} + x^{\mathrm{T}} - \frac{1}{nv(\mathscr{G})}\mathbf{1}^{\mathrm{T}} \right\|_1$$
$$= \left\| \left(d - \frac{1}{nv(\mathscr{G})} \right) e_i^{\mathrm{T}} + x^{\mathrm{T}} - \frac{1}{nv(\mathscr{G})} \left(\mathbf{1}^{\mathrm{T}} - e_i^{\mathrm{T}} \right) \right\|_1$$
$$\leqslant \left(d - \frac{1}{nv(\mathscr{G})} \right) + \left\| x^{\mathrm{T}} \right\|_1 + \left\| \frac{1}{nv(\mathscr{G})} \left(\mathbf{1}^{\mathrm{T}} - e_i^{\mathrm{T}} \right) \right\|_1$$
$$= \left(d - \frac{1}{nv(\mathscr{G})} \right) + \left(\frac{1}{v(\mathscr{G})} - d \right) + \frac{n-v(\mathscr{G})-1}{nv(\mathscr{G})}$$
$$= \frac{n-v(\mathscr{G})-2}{nv(\mathscr{G})} + \frac{1}{v(\mathscr{G})},$$

proving (2.5). The proof of (2.6) is analogous. □

We are now in a position to prove the main result of this paper.

Theorem 2.7. *Suppose that $\mathscr{G}$ is a non-complete, connected graph on n vertices with $n \geqslant v(\mathscr{G})^2$. Then $a(\mathscr{G}) = v(\mathscr{G})$ if and only if $1/\mathscr{Z}\left(L^{\#}(\mathscr{G})\right) = v(\mathscr{G})$.*

Proof. Since we know that $v(\mathscr{G}) \geqslant a(\mathscr{G}) \geqslant 1/\mathscr{Z}(L^{\#}(\mathscr{G}))$, we see that if $1/\mathscr{Z}(L^{\#}(\mathscr{G})) = v(\mathscr{G})$, then necessarily $v(\mathscr{G}) = a(\mathscr{G})$.

Now suppose that $v(\mathscr{G}) = a(\mathscr{G})$. From Theorem 2.1 we know that $\mathscr{G} = \mathscr{G}_1 \vee \mathscr{G}_2$, where $\mathscr{G}_1$ is a disconnected graph on $n - v(\mathscr{G})$ vertices with $a(\mathscr{G}_2) \geqslant 2v(\mathscr{G}) - n$ and where $\mathscr{G}_2$ is a graph on $v(\mathscr{G})$ vertices. Evidently we need only prove that $\mathscr{Z}(L^{\#}(\mathscr{G})) = 1/v(\mathscr{G})$. By Corollary 2.3 we have that

$$
\begin{aligned}
&\mathscr{Z}(L^{\#}(\mathscr{G})) \\
&\quad = \max\Bigg\{ \mathscr{Z}\left([L(\mathscr{G}_1) + v(\mathscr{G})I]^{-1}\right), \mathscr{Z}\left([L(\mathscr{G}_2) + (n - v(\mathscr{G}))I]^{-1}\right), \\
&\qquad \max_{\substack{1 \leqslant i \leqslant n - v(\mathscr{G}) \\ n - v(\mathscr{G}) + 1 \leqslant j \leqslant n}} \Bigg\{ \frac{1}{2}\Bigg[\left\| e_i^{\mathrm{T}}((L(\mathscr{G}_1) + v(\mathscr{G})I)^{-1} - \frac{1}{nv(\mathscr{G})}J) \right\|_1 \\
&\qquad\qquad + \Bigg\| e_j^{\mathrm{T}}((L(\mathscr{G}_2) + (n - v(\mathscr{G}))I)^{-1} \\
&\qquad\qquad\qquad - \frac{1}{(n - v(\mathscr{G}))n}J) \Bigg\|_1 \Bigg] \Bigg\} \Bigg\}.
\end{aligned}
$$

Note that $(L(\mathscr{G}_2) + (n - v(\mathscr{G}))I)^{-1}$ is a non-negative matrix with row sums $1/(n - v(\mathscr{G}))$, thus showing that

$$
\mathscr{Z}((L(\mathscr{G}_2) + (n - v(\mathscr{G}))I)^{-1}) \leqslant \frac{1}{n - v(\mathscr{G})} \leqslant \frac{1}{v(\mathscr{G})}.
$$

From Lemma 2.6 we find that for each pair i and j,

$$
\begin{aligned}
&\frac{1}{2}\Bigg[\left\| e_i^{\mathrm{T}}\left(L(\mathscr{G}_1) + v(\mathscr{G})I)^{-1} - \frac{1}{nv(\mathscr{G})}J \right) \right\|_1 \\
&\qquad + \left\| e_j^{\mathrm{T}}\left((L(\mathscr{G}_2) + (n - v(\mathscr{G}))I)^{-1} - \frac{1}{n(n - v(\mathscr{G}))}J \right) \right\|_1 \Bigg] \\
&\quad \leqslant \frac{1}{2}\left[\frac{n - v(\mathscr{G}) - 2}{nv(\mathscr{G})} + \frac{1}{v(\mathscr{G})} + \frac{v(\mathscr{G}) - 2}{n(n - v(\mathscr{G}))} + \frac{1}{n - v(\mathscr{G})} \right] \\
&\quad = \frac{1}{v(\mathscr{G})} + \frac{1}{2}\left[\frac{n - v(\mathscr{G}) - 2}{nv(\mathscr{G})} - \frac{1}{v(\mathscr{G})} + \frac{v(\mathscr{G}) - 2}{n(n - v(\mathscr{G}))} + \frac{1}{n - v(\mathscr{G})} \right] \\
&\quad = \frac{1}{v(\mathscr{G})} + \frac{1}{2}\left[\frac{2(v(\mathscr{G})^2 - n)}{nv(\mathscr{G})(n - v(\mathscr{G}))} \right] \leqslant \frac{1}{v(\mathscr{G})},
\end{aligned}
$$

the last inequality following from the hypothesis that $n \geqslant v(\mathscr{G})^2$. Finally, we claim that

$$
\mathscr{Z}([L(\mathscr{G}_1) + v(\mathscr{G})I]^{-1}) = \frac{1}{v(\mathscr{G})}.
$$

To see this note that $(L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}$ is a direct sum of positive matrices each of which corresponds to a connected component of $\mathscr{G}$ and each with constant row sums equal to $1/v(\mathscr{G})$. Hence

$$\left\|(e_i^{\mathrm{T}}-e_j^{\mathrm{T}})(L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}\right\|_1 \leqslant \frac{2}{v(\mathscr{G})}$$

for any pair of indices i and j. Furthermore with i and j corresponding to rows in different direct summands,

$$\left\|(e_i^{\mathrm{T}}-e_j^{\mathrm{T}})(L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}\right\|_1 = \frac{2}{v(\mathscr{G})}.$$

Hence

$$\mathscr{Z}((L(\mathscr{G}_1)+v(\mathscr{G})I)^{-1}) = \frac{1}{v(\mathscr{G})},$$

as claimed. It now follows that

$$\mathscr{Z}(L^{\#}(\mathscr{G})) = \frac{1}{v(\mathscr{G})}. \qquad \square$$

Theorem 2.7 includes the hypothesis that $n \geqslant v^2(\mathscr{G})$. The following example shows that that hypothesis cannot be relaxed.

Example 2.8. Suppose that we have integers n and w such that $w^2 > n$ and $n/2 \geqslant w \geqslant 3$. Consider the graph $\mathscr{G}$ on n vertices constructed as follows: Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be empty graphs on $n-w$ and w vertices, respectively, and let $\mathscr{G}=\mathscr{G}_1 \vee \mathscr{G}_2$. Since $w \leqslant n/2$, we find that $v(\mathscr{G})=w$. Further, we have that $a(\mathscr{G}_2)=0 \geqslant 2v(\mathscr{G})-n$ so that, by Theorem 2.1, $a(\mathscr{G})=v(\mathscr{G})$. By Lemma 2.2, we see that

$$L^{\#}(\mathscr{G}) = \left[\begin{array}{c|c} \frac{1}{v(\mathscr{G})}I-\left[\frac{n+v(\mathscr{G})}{v(\mathscr{G})n^2}\right]J & -\frac{1}{n^2}J \\ \hline -\frac{1}{n^2}J & \frac{1}{n-v(\mathscr{G})}I-\left[\frac{2n-v(\mathscr{G})}{(n-v(\mathscr{G}))n^2}\right]J \end{array}\right],$$

where the (1, 1)-block is an $(n-v(\mathscr{G}))\times(n-v(\mathscr{G}))$ matrix and the (2, 2)-block is a $v(\mathscr{G})\times v(\mathscr{G})$ matrix. It follows that

$$\begin{aligned}\mathscr{Z}(L^{\#}(G)) &= \frac{1}{2}\Bigg[\left\|\frac{1}{v(\mathscr{G})}e_i^{\mathrm{T}}-\frac{1}{nv(\mathscr{G})}\mathbf{1}^{\mathrm{T}}\right\|_1 \\ &\qquad + \left\|\frac{1}{n-v(\mathscr{G})}e_i^{\mathrm{T}}-\frac{1}{n(n-v(\mathscr{G}))}\mathbf{1}^{\mathrm{T}}\right\|_1\Bigg] \\ &= \frac{1}{2}\left[\frac{1}{v(\mathscr{G})}+\frac{n-v(\mathscr{G})-2}{nv(\mathscr{G})}+\frac{1}{n-v(\mathscr{G})}+\frac{v(\mathscr{G})-2}{n(n-v(\mathscr{G}))}\right] \\ &= \frac{1}{v(\mathscr{G})}+\frac{v(\mathscr{G})^2-n}{nv(\mathscr{G})(n-v(\mathscr{G}))} \; > \; \frac{1}{v(\mathscr{G})}.\end{aligned}$$

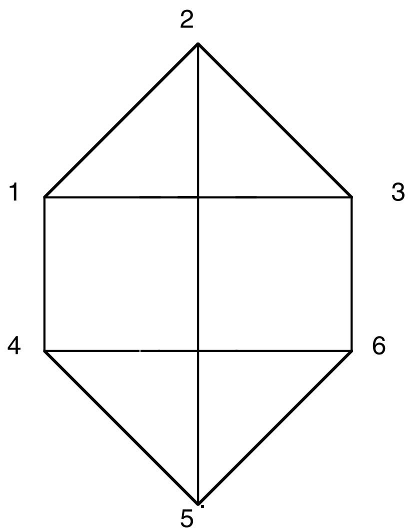
We see then that, in general, the hypothesis in Theorem 2.7 that $n \geqslant v(\mathscr{G})^2$ cannot be weakened.

We close the paper with an example showing that, in general, $1/\mathscr{Z}(L^{\#}(\mathscr{G})) = a(\mathscr{G})$ does not imply that $1/\mathscr{Z}(L^{\#}(\mathscr{G})) = a(\mathscr{G}) = v(\mathscr{G})$.

Example 2.9. Let $\mathscr{G}_k$ be the graph on $n = 2k$, $k > 1$, vertices, whose Laplacian is given by

$$L = \begin{bmatrix} (k+1)I_k - J_k & -I_k \\ -I_k & (k+1)I_k - J_k \end{bmatrix}.$$

For instance, $\mathscr{G}_3$ is the following graph on six vertices:



The eigenvalues of L turn out to be $0, 2, k$, and $k+2$, each of the last two with multiplicity $k-1$; in particular, we find that $a(\mathscr{G}_k) = 2$. It is also straightforward to determine that $v(\mathscr{G}_k) = k$

It turns out that $L^{\#}$ is given by

$$L^{\#} = \begin{bmatrix} \frac{k+1}{k(k+2)} I_k + \frac{(k+1)(k-2)}{4k^2(k+2)} J_k & \frac{1}{k(k+2)} I_k - \frac{k^2+k+2}{4k^2(k+2)} J_k \\ \frac{1}{k(k+2)} I_k - \frac{k^2+k+2}{4k^2(k+2)} J_k & \frac{k+1}{k(k+2)} I_k + \frac{(k+1)(k-2)}{4k^2(k+2)} J_k \end{bmatrix} - \frac{1}{4k^2} J_{2k}.$$

From this last expression, it is not difficult to determine that $1/\mathscr{Z}(\mathscr{G}_k) = 2$, so that when $k \geqslant 3$, we have $1/\mathscr{Z}(\mathscr{G}_k) = a(\mathscr{G}_k) = 2 < v(\mathscr{G}_k) = k$.

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