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Zp-Modules with Partial Decomposition Bases in $L^{\delta}_{\infty\omega}$

Carol Jacoby

Peter Loth
Sacred Heart University, lothp@sacredheart.edu

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$\mathbb{Z}_p$-MODULES WITH PARTIAL DECOMPOSITION BASES IN $L^{\delta}_{\infty\omega}$

CAROL JACOBY AND PETER LOTH

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Abstract. We consider the class of mixed $\mathbb{Z}_p$-modules with partial decomposition bases. This class includes those modules classified by Ulm and Warfield and is closed under $L_{\infty\omega}$-equivalence. In the context of $L_{\infty\omega}$-equivalence, Jacoby defined invariants for this class and proved a classification theorem. Here we examine this class relative to $L^{\delta}_{\infty\omega}$, those formulas of quantifier rank $\leq$ some ordinal δ, defining invariants and proving a classification theorem. This generalizes a result of Barwise and Eklof.

1. Introduction

Ulm's Theorem [U] presents invariants that classify countable torsion abelian groups up to isomorphism. Barwise and Eklof [BE] extended this result to the classification of arbitrary torsion abelian groups up to equivalence in the infinitary language $L_{\infty\omega}$. Warfield [W2] extended Ulm's Theorem by classifying a certain class of mixed local groups, which have come to be called Warfield groups, up to isomorphism. The first author [J2] defined a class of mixed $\mathbb{Z}_p$-modules which includes those studied by Warfield and is closed under $L_{\infty\omega}$-equivalence. The defining property of these modules is the existence of a partial decomposition basis, a generalization of the concept of decomposition basis. In the context of $L_{\infty\omega}$-equivalence, invariants were defined for this class and a classification theorem was proved.

Here we look at this class in the context of $L^{\delta}_{\infty\omega}$ for some ordinal δ, i.e, the formulas of quantifier rank $\leq \delta$. These results were needed to prove theorems

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about definability and expressibility of the invariants and the class [J3]. Classification of this class in $L^{\delta}_{\infty\omega}$ was included in [J2], but recently the second author noted shortcomings in some of these proofs. These have been corrected here. In addition, new results and approaches have streamlined many of the theorems. We focus on local groups, i.e., modules over $\mathbb{Z}_p$. The global case is considered in [JLLS] and [JL].

Section 2 presents the definitions and results from algebra that will be needed. Section 3 defines $L^{\delta}_{\infty\omega}$ and presents Karp's Theorem that unites the algebraic and model theoretic views. Section 4 proves an extension result that will be needed in the classification theorem and Section 5 proves the classification theorem for modules with partial decomposition bases in $L^{\delta}_{\infty\omega}$.

For notation and terminology on abelian groups and on model theory, we may refer to the books [F1], [F2], [L] and [R].

2. Algebraic background

All modules considered in this paper are modules over $\mathbb{Z}_p$, the ring of integers localized at the prime p. For every ordinal α, a submodule $p^\alpha M$ of the module M is defined as follows: $pM = \{px : x \in M\}$, $p^{\alpha+1}M = p(p^\alpha M)$, and $p^\alpha M = \bigcap_{\beta<\alpha} p^\beta M$ if α is a limit ordinal. The *length* of M is the least ordinal τ such that $p^\tau M = p^{\tau+1}M$. The *height* of an element $x \in M$, written $|x|$ or $|x|_M$, is α if $x \in p^\alpha M \setminus p^{\alpha+1}M$ and $|x| = \infty$ if $x \in p^\infty M = \bigcap_\alpha p^\alpha M$. The module $p^\infty M$ is called the *divisible part of* M and M is called *reduced* if $p^\infty M = 0$. The submodule of M generated by a subset S of M is denoted by $\langle S\rangle$, and $\langle S\rangle^0$ is the set of all elements $x \in M$ such that $rx \in \langle S\rangle$ for some $0 \neq r \in \mathbb{Z}_p$. Notice that the height of $x \in S$ computed in M coincides with the height of x computed in $\langle S\rangle^0$. A submodule H of M is called *nice* (*in* M) if

$$p^\alpha(M/H) = (p^\alpha M + H)/H$$

for all ordinals α. Note that H is nice if and only if every coset of H has an element $x \in M$ which is *proper with respect to* H, that is, if x has maximal height among all elements in the coset $x + H$ (cf. [L, Proposition 1.4]). In this case, we have $|x + h| = \min\{|x|, |h|\}$ for all $h \in H$. The torsion part of M is denoted by tM and we let $M[p] = \{x \in M : px = 0\}$. Then the *Ulm-Kaplansky invariants* of a module M are defined by

$$u(\alpha, M) = \dim\,(p^\alpha M)[p]/(p^{\alpha+1}M)[p]$$

where α is an ordinal, and

$$u(\infty, M) = \dim\,(p^\infty M)[p].$$

Ulm [U] proved that these cardinal numbers are isomorphism invariants for countable torsion modules.

An *Ulm sequence* is a sequence $(\beta_i : i < \omega)$ where each β_i is an ordinal or the symbol ∞ such that $\beta_i < \beta_{i+1}$ for all i and we use the convention $\alpha < \infty$ whenever α is an ordinal or the symbol ∞. Two Ulm sequences $\overline{\beta} = (\beta_i : i < \omega)$ and $\overline{\gamma} = (\gamma_i : i < \omega)$ are called *equivalent*, and we write $\overline{\beta} \sim \overline{\gamma}$, if there exist $m, n < \omega$ such that $(\beta_{i+m} : i < \omega) = (\gamma_{i+n} : i < \omega)$. The *Ulm sequence of* $x \in M$ is the sequence $U(x) = (|p^i x| : i < \omega)$.

A subset $X = \{x_i : i \in I\}$ of M is called a *decomposition set* if all elements of X are independent and have infinite order such that

$$|r_1x_1 + \ldots + r_nx_n| = \min\{|r_1x_1|, \ldots, |r_nx_n|\}$$

for all $x_1, \ldots, x_n \in X$ and $r_1, \ldots, r_n \in \mathbb{Z}_p$. If in addition $M/\langle X\rangle$ is torsion, X is called a *decomposition basis for* M. More generally, a system $\mathcal{C}$ is called a *partial decomposition basis for* M if

(1) $\mathcal{C}$ is a non-empty collection of finite subsets of M;
(2) if $X \in \mathcal{C}$, then X is a decomposition set;
(3) if $X \in \mathcal{C}$ and $x \in M$, there is $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $x \in \langle Y\rangle^0$.

It is clear that if X is a decomposition basis for M, then the collection of all finite subsets of X is a partial decomposition basis for M. *Warfield modules* are modules M possessing a decomposition basis X such that $\langle X\rangle$ is nice and $M/\langle X\rangle$ is simply presented. Their *Warfield invariants* are defined by

$$w(e, M) = |\{x \in X : U(x) \in e\}|$$

where e is an equivalence class of Ulm sequences. Warfield [W2] showed that these invariants, together with the Ulm-Kaplansky invariants, form a complete set of isomorphism invariants for Warfield modules.

3. Model-theoretic preliminaries

$L_{\infty\omega}$ is an extension of a language of first order logic to allow conjunctions and disjunctions over arbitrary sets of formulas and quantifications over finite sets of variables. $L^\delta_{\infty\omega}$ consists of the formulas of this language with quantifier rank no more than δ (cf. [BE]). We say models $\mathfrak{A}$ and $\mathfrak{B}$ are $L^\delta_{\infty\omega}$-*equivalent*, written $\mathfrak{A} \equiv_\delta \mathfrak{B}$ if they satisfy the same sentences of $L^\delta_{\infty\omega}$. In particular, $\mathfrak{A} \equiv_\infty \mathfrak{B}$ if they satisfy the same sentences of $L_{\infty\omega}$. $L^\delta_{\infty\omega}$-equivalence can be characterized by partial isomorphisms having the back-and-forth property:

Theorem 3.1 ([Kar]). *Let $\mathfrak{A} = \langle A, \ldots \rangle$ and $\mathfrak{B} = \langle B, \ldots \rangle$ be models for $L_{\infty\omega}$ and δ an ordinal or the symbol ∞. Then the following are equivalent:*

(i) $\mathfrak{A} \equiv_\delta \mathfrak{B}$;

(ii) *For each ordinal $\nu \leq \delta$ there is a non-empty set I_ν of isomorphisms on finitely generated substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that*

(a) if $\nu \leq \mu$, then $I_\mu \subseteq I_\nu$;

(b) if $\nu < \delta$, $f \in I_{\nu+1}$ and $x \in A$ ($y \in B$, resp.), then f extends to a map $f' \in I_\nu$ such that $x \in domain(f')$ ($y \in range(f'), resp.$).

If $\delta = \infty$, the sets I_ν in (ii) can be chosen to be all equal to some fixed set I.

4. Extending α-height-preserving isomorphisms

Let S and T be submodules of modules M and N, respectively, and let α be an ordinal or the symbol ∞. Then we say that an isomorphism $f : S \to T$ *preserves heights up to* α if

$$\min\{|x|, \alpha\} = \min\{|f(x)|, \alpha\}$$

for all $x \in S$ where all heights are computed in M and N, respectively. In this case, f is also called *α-height-preserving* (or *height-preserving* if $\alpha = \infty$).

The following result is a special case of [J1, Theorem 4.2] and uses modifications of the Ulm-Kaplansky invariants (see [BE] and [J1]): for a module M and an ordinal α, let $\hat{u}(\alpha, M) = \min\{u(\alpha, M), \omega\}$ and $\hat{u}(\infty, M) = \min\{u(\infty, M), \omega\}$.

Theorem 4.1 ([J1]). *Let M and N be modules, α an ordinal and $f : S \to T$ an α-height-preserving isomorphism where S and T are finitely generated submodules of M and N, respectively. Suppose that $\hat{u}(\sigma, M) \leq \hat{u}(\sigma, N)$ for all $\sigma < \alpha$. If $x \in M$ is proper with respect to S, $px \in S$ and $|x|+1 < \alpha$, then for a suitable y in N, f can be extended to an α-height-preserving isomorphism $g : \langle S, x \rangle \to \langle T, y \rangle$ by mapping x onto y.*

The following lemmas will be useful.

Lemma 4.2 ([GLLS], [J2]). *Let M and N be modules, α an ordinal and $f : S \to T$ an α-height-preserving isomorphism where S and T are submodules of M and N, respectively. Suppose that $x \in M$, $y \in N$, $|x| \geq \alpha$ and $|y| \geq \alpha$. Suppose further that $x + S$ and $y + T$ have the same order and if p^n is the order of $x + S$, then $f(p^n x) = p^n y$. Then f can be extended to an α-height-preserving isomorphism $f' : \langle S, x \rangle \to \langle T, y \rangle$ by mapping x onto y.*

Lemma 4.3 ([J1]). *Let M be a reduced module of finite rank with a decomposition basis $\{x_1, \ldots, x_n\}$. Then there is a torsion module T and submodules M_i of M $(i = 1, \ldots, n)$ such that*

$$M \oplus T \equiv_\infty M_1 \oplus \cdots \oplus M_n$$

and $x_i \in M_i$ for all $i = 1, \ldots, n$. In fact, the set I of all height-preserving isomorphisms between finitely generated submodules of $M \oplus T$ and $M_1 \oplus \cdots \oplus M_n$ extending the canonical map $\varphi : \langle x_1, \ldots, x_n\rangle \to \langle x_1, \ldots, x_n\rangle$ satisfies the conditions of Karp's Theorem 3.1.

Lemma 4.4. *Let M be a module, X a decomposition basis for M and S a finitely generated submodule of M such that $S \cap \langle X\rangle = \langle S \cap X\rangle$. If $y \in X$ such that $y \notin S$ then there is an $n \in \omega$ such that*

$$|rp^n y + s| = \min\{|rp^n y|, |s|\}$$

for all $r \in \mathbb{Z}_p$ and $s \in S$.

Proof. It is easy to verify that $S \cap \langle y\rangle = 0$. Since S is finitely generated, $S \cap X$ is finite, say $S \cap X = \{x_1, \ldots, x_m\}$ where the first k elements are exactly those whose Ulm sequence is not equivalent to $(\infty, \infty, \ldots)$ $(0 \le k \le m)$. Clearly $\{y, x_1, \ldots, x_m\}$ is a decomposition basis for

$$N = (S \oplus \langle y\rangle)^0.$$

If $U(y) \sim (\infty, \infty, \ldots)$ there is $n \in \omega$ such that $|p^n y| = \infty$, so the claim follows immediately. Suppose $U(y) \not\sim (\infty, \infty, \ldots)$. Letting D be the divisible part of N we have $\langle y, x_1, \ldots, x_k\rangle \cap D = 0$. By [Kap, Theorem 6], the projection $\langle y, x_1, \ldots, x_k\rangle \oplus D \to D$ can be extended to N resulting in $N = R \oplus D$ where R is a reduced module with decomposition basis $\{y, x_1, \ldots, x_k\}$. By Lemma 4.3 we have

$$R \oplus T \equiv_\infty N_0 \oplus \ldots \oplus N_k$$

where T is torsion, $y \in N_0$ and $x_i \in N_i$ for $i = 1, \ldots, k$. Let I be the set of partial isomorphisms as in Lemma 4.3. For a map $f : A \to B$ in I define

$$\begin{array}{rccc} f' : & A \oplus D & \to & B \oplus D. \\ & (a, x) & \mapsto & (f(a), x) \end{array}$$

Then $I' = \{f' : f \in I\}$ is a set of isomorphisms between submodules of $N \oplus T$ and $N_0 \oplus \ldots \oplus N_k \oplus D$ satisfying the conditions of Karp's Theorem 3.1, hence the finitely generated submodule $S \oplus \langle y\rangle$ of N is contained in the domain of some $f' \in I'$. If $\pi : N_0 \oplus \cdots \oplus N_k \oplus D \to N_0$ is the projection, then $\pi f'(S)$ is finitely generated since S is. It is also torsion, since if $z \in S$, then there is a nonzero

$r \in \mathbb{Z}_p$ such that $rz \in S \cap \langle X \rangle = \langle S \cap X \rangle$, so by the construction of f and f' we have $f'(rz) = rz \in N_1 \oplus \ldots \oplus N_k \oplus D$ which implies $r\pi f'(z) = \pi f'(rz) = 0$. But then the elements of $\pi f'(S)$ have only a finite number of heights. Since $U(y) \not\sim (\infty, \infty, \ldots)$ and f' is height-preserving, the elements $p^n f'(y)$ $(n \in \omega)$ have infinitely many heights, so there is an $n \in \omega$ such that $|rp^n y| \neq |s|$ for all $0 \neq r \in \mathbb{Z}_p$ and $s \in \pi f'(S)$.

Now let $s \in S$, $r \in \mathbb{Z}_p$ and write $f'(s) = s_0 + \cdots + s_k + d$ where $s_i \in N_i$ for $i = 0, \ldots, k$ and $d \in D$. Then

$$\begin{aligned}|rp^n y + s| &= |f'(rp^n y + s)| = |f'(rp^n y) + s_0 + \cdots + s_k + d| \\ &= \min\{|f'(rp^n y) + s_0|, |s_1|, \ldots, |s_k|, |d|\} \\ &= \min\{|f'(rp^n y)|, |s_0|, \ldots, |s_k|, |d|\} \\ &= \min\{|f'(rp^n y)|, |f'(s)|\} \\ &= \min\{|rp^n y|, |s|\}\end{aligned}$$

□

Lemma 4.5 ([W1]). *Suppose N and S are submodules of a module M such that N is nice in M and S contains N such that S/N is torsion and finitely generated. Then S is nice in M.*

The following extension result will be needed in Section 5.

Lemma 4.6. *Let M and N be modules such that $\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ for all $\alpha < \omega(\nu + 1)$ for some ordinal ν and if $length(tM) < \omega(\nu + 1)$, then $\hat{u}(\infty, M) = \hat{u}(\infty, N)$. Suppose $f : S \to T$ is an isomorphism which preserves heights up to $\omega\nu + k + n + 1$ for some $k, n \in \omega$, where S and T are nice, finitely generated submodules of M and N respectively. Let $a \in M$ and suppose*

(1) *$p^{n+1}a \in S$ and $p^n a \notin S$;*
(2) *for all $0 \leq m \leq n$ the following is true: if for all $x \in p^m a + \langle S, p^{m+1}a \rangle$ we have $|x| < \omega(\nu + 1)$, then $|x| + 1 < \omega\nu + k$ for all such x;*
(3) *if $length(tM) < \omega(\nu + 1)$ then $length(tM) < \omega\nu + k$.*

Then f extends to an $\omega\nu + k$-height-preserving isomorphism $g : \langle S, a \rangle \to T'$ for some submodule T' of N.

Proof. Case 1: First consider the case in which for every $0 \leq m \leq n$ and $x \in p^m a + \langle S, p^{m+1}a \rangle$ we have $|x| < \omega(\nu + 1)$. We will prove by induction on m, $0 \leq m \leq n + 1$, that f extends to an $\omega\nu + k$-height-preserving isomorphism f_m such that $domain(f_m) = \langle S, p^{n+1-m}a \rangle$. Let $f_0 = f$. Suppose f_m has been chosen. Let $y = p^{n-m}a$. Then $py \in S_m = domain(f_m)$. If $y \in S_m$, we may let

$f_{m+1} = f_m$. So suppose $y \notin S_m$. For all $x \in y + S_m = p^{n-m}a + \langle S, p^{n+1-m}a\rangle$ we have $|x| < \omega(\nu+1)$ by assumption so $|x|+1 < \omega\nu+k$ by (2). By Lemma 4.5, S_m is nice in M, so we may choose x proper with respect to S_m. Then by Theorem 4.1, f_m extends to an $\omega\nu+k$-height-preserving isomorphism f_{m+1} such that

$$domain(f_{m+1}) = \langle S_m, x\rangle = \langle S_m, y\rangle = \langle S, p^{n+1-m}a, p^{n-m}a\rangle = \langle S, p^{n-m}a\rangle.$$

This completes the induction. Let $g = f_{n+1}$.

Case 2: Now suppose for some $0 \le m \le n$ there is an $x \in p^m a + \langle S, p^{m+1}a\rangle$ such that $|x| \ge \omega(\nu+1)$. We may assume that x has been chosen so that m is minimal. We will prove that f extends to x.

Choose j least such that $p^{j+1}x \in S$. Then

$$|p^{j+1}x| \ge |x| + j + 1 \ge \omega(\nu+1) + j + 1 > \omega\nu + m + k + j + 1,$$

so $|f(p^{j+1}x)| \ge \omega\nu + m + k + j + 1$ since $j \le n - m$ and f preserves heights up to $\omega\nu + k + n + 1$. Thus we may choose $b_0 \in N$ such that $p^{j+1}b_0 = f(p^{j+1}x)$ and $|b_0| \ge \omega\nu + m + k$. We claim there is a $b \notin T$ with these same properties. If $p^j b_0 \notin T$, we may let $b = b_0$. Suppose $p^j b_0 \in T$, say $p^j b_0 = f(a_0)$ for some $a_0 \in S$.

Case 2a Suppose first that $length(tM) < \omega(\nu+1)$. Then $length(tM) < \omega\nu+k$ by (3). Since

$$f(p(p^j x - a_0)) = p^{j+1}b_0 - p^{j+1}b_0 = 0,$$

we have $p(p^j x - a_0) = 0$. Also $|p^j x - a_0| \ge \omega\nu + m + k$ and $p^j x - a_0 \ne 0$ since $p^j x \notin S$. Let $y = p^j x - a_0$. Suppose $|y| \ne \infty$. Since $y \in tM$, $\infty \ne |y| \ge \omega\nu+m+k$ contradicts $length(tM) < \omega\nu + k$. Consequently, we have $|y| = \infty$. But then

$$\begin{aligned}\hat{u}(\infty, N) = \hat{u}(\infty, M) &= min\{\dim(p^\infty M)[p], \omega\} \ge \dim(p^\infty M \cap \langle S, y\rangle)[p]\\ &= \dim\langle y\rangle \oplus (S[p] \cap p^\infty M),\end{aligned}$$

the latter equality following from the modular law. Letting $\alpha = \omega\nu + k + n + 1$, we have

$$\begin{aligned}\dim\langle y\rangle \oplus (S[p] \cap p^\infty M) &> \dim S[p] \cap p^\infty M = \dim S[p] \cap p^\infty(tM)\\ &= \dim S[p] \cap p^\alpha(tM) = \dim S[p] \cap p^\alpha M\\ &= \dim T[p] \cap p^\alpha N\end{aligned}$$

since f is α-height-preserving. Then there is an element in $(p^\infty N)[p]$ that is not in $T[p] \cap p^\alpha N$, hence it is not in T. Write it as $p^j b_1$ where $|b_1| = \infty$.

It is easy to verify that $b = b_0 + b_1$ satisfies the desired condition.

Case 2b Now suppose $length(tM) \ge \omega(\nu+1)$. Then $length(tN) \ge \omega(\nu+1)$ since otherwise the modified Ulm invariants would disagree between $length(tN)$

and $\omega(\nu+1)$. Again, we will choose an appropriate b as the image of x. Barwise and Eklof [BE, (2.6) and (2.7)] proved that if G is a p-group, S a subgroup of G of finite rank, α a limit ordinal $\leq length(G)$ and $\beta < \alpha$, then for any $n \in \omega$ there exists a $z \in p^\beta G$ such that $p^n z \notin S$ and $p^{n+1}z = 0$. The proof applies equally well to $\mathbb{Z}_p$-modules, so by applying it to tN, the limit ordinal $\omega(\nu+1) \leq length(tN)$ and $\beta = \omega\nu + m + k$, we get $b_1 \in p^{\omega\nu+m+k}N$ such that $p^j b_1 \notin T$ and $p^{j+1}b_1 = 0$. Let $b = b_0 + b_1$.

In either case we have chosen a b such that

$$|b| \geq \omega\nu + m + k, p^j b \notin T \text{ and } p^{j+1}b = f(p^{j+1}x).$$

Define $f' : \langle S, x\rangle \to \langle T, b\rangle$ as in Lemma 4.2. Then f' preserves heights up to $\omega\nu + m + k$.

We have shown that f can be extended to an $\omega\nu + m + k$-height-preserving map f' with x in its domain. Now we must show that f' can be extended to include a. We claim that $\langle S, x\rangle = \langle S, p^m a\rangle$. Recall that $x \in p^m a + \langle S, p^{m+1}a\rangle$, say $x = p^m a + rp^{m+1}a + s$ for some $r \in \mathbb{Z}_p$ and $s \in S$. Let $y = p^m a + rp^{m+1}a = p^m(1+rp)a$. Then $x - y \in S$, so $\langle S, x\rangle = \langle S, y\rangle$. But $1 + rp$ and p are relatively prime, so $\langle y\rangle = \langle p^m a\rangle$ and thus $p^m a \in \langle S, x\rangle$. Thus $\langle S, x\rangle = \langle S, p^m a\rangle$.

Now let $S' = \langle S, p^m a\rangle = domain(f')$. Then $p^m a \in S'$ and for every $l \leq m - 1$ and every

$$z \in p^l a + \langle S', p^{l+1}a\rangle = p^l a + \langle S, p^{l+1}a\rangle,$$

$|z| < \omega(\nu+1)$ by the minimality of m. Thus by Case 1 applied to $f' : S' \to T'$ and $m - 1$, we may extend f' to include a such that the extension is $\omega\nu + k$-height-preserving. □

Corollary 4.7. *Let M and N be modules such that $\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ for all α and $\hat{u}(\infty, M) = \hat{u}(\infty, N)$. Suppose $f : S \to T$ is a height-preserving isomorphism where S and T are nice, finitely generated submodules of M and N respectively. If $a \in M$ and $p^r a \in S$ for some $r \in \omega$, then f extends to a height-preserving isomorphism $g : S' = \langle S, a\rangle \to T'$ for some submodule T' of N.*

Proof. Let ν be an ordinal such that $length(M) < \omega\nu$ and apply Lemma 4.6. □

Corollary 4.8 ([BE])**.** *Let G and H be p-groups and δ an ordinal such that*

(1) $\hat{u}(\alpha, G) = \hat{u}(\alpha, H)$ *for all* $\alpha < \omega\delta$;

(2) *if* $length(G) < \omega\delta$, *then* $\hat{u}(\infty, G) = \hat{u}(\infty, H)$.

Then $G \equiv_\delta H$.

Proof. Consider the system $\{I_\nu : \nu \leq \delta\}$ where each I_ν consists of all $\omega\nu$-height-preserving isomorphisms $f : S \to T$ where S and T are finite subgroups of G and

H, respectively. Let $f \in I_{\nu+1}(\nu < \delta)$ and $a \in G$. Since S is finite, there are $k, n \in \omega$ such that conditions (1)-(3) of Lemma 4.6 hold, hence f extends to $g \in I_\nu$ with $a \in domain(g)$. By symmetry, condition (ii) of Karp's Theorem 3.1 is satisfied, hence $G \equiv_\delta H$. □

5. Classification in $L^\delta_{\infty\omega}$

Let α be an ordinal. We call two Ulm sequences (β_i) and (γ_i) *equal up to* α if

$$\min\{\beta_i, \alpha\} = \min\{\gamma_i, \alpha\}$$

for all $i < \omega$. In this case we write $(\beta_i) =_\alpha (\gamma_i)$. Two equivalence classes e and e' of Ulm sequences are called α-*equivalent*, and we write $e \sim_\alpha e'$, if there are Ulm sequences $(\beta_i) \in e$ and $(\gamma_i) \in e'$ which are equal up to α.

For a module M with partial decomposition basis $\mathcal{C}$, define $\hat{w}(e, M)$ to be the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ such that $U(x_i) \in e$ for all $i = 1, \ldots, n$. If no such n exists, put $\hat{w}(e, M) = \omega$. If α is an ordinal, define

$$\hat{w}_\alpha(e, M) = \min\{\sum_{e' \sim_a e} \hat{w}(e', M), \omega\}.$$

Note that if X is a finite decomposition set, then for any e and any α,

$$\hat{w}_\alpha(e, \langle X \rangle^0) = \sum_{e' \sim_\alpha e} |\{x \in X : U(x) \in e'\}| = |\{x \in X : [U(x)] \sim_\alpha e\}|.$$

Lemma 5.1. *Let M be a module with partial decomposition basis $\mathcal{C}$. If $X \in \mathcal{C}$, then $\hat{w}_\alpha(e, M) \geq \hat{w}_\alpha(e, \langle X \rangle^0)$ for any ordinal α and equivalence class e of Ulm sequences.*

Proof. The set X is a decomposition basis for $\langle X \rangle^0$. Then $\hat{w}(e', M) \geq \hat{w}(e', \langle X \rangle^0)$ for all equivalence classes e', therefore $\hat{w}_\alpha(e, M) \geq \hat{w}_\alpha(e, \langle X \rangle^0)$. □

Theorem 5.2. *Let M be a module with partial decomposition basis $\mathcal{C}$, α an ordinal and n a positive integer. Suppose $\hat{w}_\alpha(e, M) \geq n$ and $X \in \mathcal{C}$. Then there is an $X' \in \mathcal{C}$ such that $X \subseteq X'$ and X' has $\geq n$ elements x with $[U(x)] \sim_\alpha e$.*

Proof. There is a finite set $\{e_1, \ldots, e_m\}$ of distinct equivalence classes of Ulm sequences $\sim_\alpha e$ such that $\sum_{i=1}^m w(e_i, M) \geq n$. If $w(e_i, M) \geq \omega$ for some i we are done, so assume that $w(e_i, M) < \omega$ for all $i = 1, \ldots, m$.

For each $i = 1, \ldots, m$ we let $n_i = w(e_i, M)$ and define inductively $X_i \in \mathcal{C}$ having $\geq n_i$ elements x with $U(x) \in e_i$ such that $X \subseteq X_1 \subseteq \ldots \subseteq X_i$. Then it is clear that $X' = X_m$ satisfies the required properties. Now the induction can be easily carried out since by [J1, Theorem 3.6], for any $Y \in \mathcal{C}$ and $i \leq m$ there exists an $X_i \in \mathcal{C}$ containing Y and having $\geq n_i$ elements x with $U(x) \in e_i$. □

Corollary 5.3. *Let M be a module with partial decomposition basis $\mathcal{C}$, α an ordinal and e an equivalence class of Ulm sequences. Then $\hat{w}_\alpha(e, M)$ is the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ satisfying $[U(x_i)] \sim_\alpha e$ for all $i = 1, \ldots, n$. If no such n exists, then $\hat{w}_\alpha(e, M) = \omega$.*

Proof. If there is a largest integer n such that there is an $X \in \mathcal{C}$ containing n elements x satisfying $[U(x)] \sim_\alpha e$, then $\hat{w}_\alpha(e, M) \leq n$ by Theorem 5.2. On the other hand, $\hat{w}_\alpha(e, M) \geq \hat{w}_\alpha(e, \langle X \rangle^0) = n$ by Lemma 5.1. If no such n exists, $\hat{w}_\alpha(e, M) = \omega$ by Lemma 5.1. □

The following facts about partial decomposition bases will be needed.

Lemma 5.4. *Let M be a module with partial decomposition basis $\mathcal{C}$ and $Y \subseteq X \in \mathcal{C}$. Then $\langle Y \rangle$ is nice in M.*

Proof. Let $x \in M$. Choose $\tilde{X} \in \mathcal{C}$ such that $X \subseteq \tilde{X}$ and $x \in \langle \tilde{X} \rangle^0$. Then $\tilde{X}$ is a decomposition basis for the module $\langle \tilde{X} \rangle^0$ and Y is a finite subset of $\tilde{X}$. By [HR, Lemma 8.1], a subgroup generated by a finite subset of a decomposition basis is nice, so $\langle Y \rangle$ is nice in $\langle \tilde{X} \rangle^0$. Since $x \in \langle \tilde{X} \rangle^0$, this means $x + \langle Y \rangle$ has an element of maximal height. Since $x \in M$ was arbitrary and the heights in M and in $\langle \tilde{X} \rangle^0$ are equal, this proves that $\langle Y \rangle$ is nice in M. □

Lemma 5.5 ([J1])**.** *Suppose M is a module with partial decomposition basis. Then M also has a partial decomposition basis $\mathcal{C}$ such that $\emptyset \in \mathcal{C}$ and for any $X \in \mathcal{C}$, $x_1, \ldots, x_n \in X$ and nonzero $a_1, \ldots, a_n \in \mathbb{Z}_p$, $\{a_1 x_1, \ldots, a_n x_n\} \in \mathcal{C}$.*

Now we are ready to prove the main result of this paper.

Theorem 5.6. *Let M and N be modules with partial decomposition bases. Let δ be an ordinal such that*

(1) *$\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ for all $\alpha < \omega\delta$;*
(2) *$\hat{w}_{\omega(\nu+1)}(e, M) = \hat{w}_{\omega(\nu+1)}(e, N)$ for all $\nu < \delta$ and equivalence class e of Ulm sequences;*
(3) *if $length(tM) < \omega\delta$ then $\hat{u}(\infty, M) = \hat{u}(\infty, N)$.*

Then $M \equiv_\delta N$.

Proof. Let $\mathcal{C}$ and $\mathcal{C}'$ be partial decomposition bases of M and N as in Lemma 5.5. For $\nu \leq \delta$ we define I_ν to be the set of all maps $f : S \to T$ such that there are $X \in \mathcal{C}$, $Y \in \mathcal{C}'$ with $f(X) = Y$ satisfying:

(i) S and T are finitely generated submodules of M and N, respectively;
(ii) f is an $\omega\nu$-height-preserving isomorphism;
(iii) $X \subseteq S \subseteq \langle X\rangle^0$ and $Y \subseteq T \subseteq \langle Y\rangle^0$.

It suffices to prove that the system $\{I_\nu : \nu \leq \delta\}$ satisfies condition (ii) of Theorem 3.1. First we note that for any ν, $I_\nu \neq \emptyset$ since the zero function is in I_ν with $S = \{0\}, T = \{0\}, X = \emptyset$ and $Y = \emptyset$.

Let $f : S \to T$ be a map in $I_{\nu+1}$ ($\nu < \delta$) with associated $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, and let $a \in M$. Then there is some $X' \in \mathcal{C}$ such that $a \in \langle X'\rangle^0$ and $X \subseteq X'$, say $X' = X \cup \{x_1, \dots, x_m\}$. For $i = 1, \dots, m$ we will define by induction $y_i \in N$ and $m_i, n_i \in \omega$ and isomorphisms

$$f_i : \langle S, p^{m_1}x_1, \dots, p^{m_i}x_i\rangle \to \langle T, p^{n_1}y_1, \dots, p^{n_i}y_i\rangle$$

that extend f and satisfy conditions (i)-(iii) of $I_{\nu+1}$ with associated decomposition sets $X_i = X \cup \{p^{m_1}x_1, \dots, p^{m_i}x_i\} \in \mathcal{C}$ and $Y_i = Y \cup \{p^{n_1}y_1, \dots, p^{n_i}y_i\} \in \mathcal{C}'$, where $f(p^{m_j}x_j) = p^{n_j}y_j$ for all $j = 1, \dots, i$. Let $f_0 = f$, $X_0 = X$ and $Y_0 = Y$. Assume this has been done for $i < m$. Let $e = [U(x_{i+1})]$. Then

$$\begin{aligned}\hat{w}_{\omega(\nu+1)}(e, N) &= \hat{w}_{\omega(\nu+1)}(e, M) \geq \hat{w}_{\omega(\nu+1)}(e, \langle X'\rangle^0)\\ &> \hat{w}_{\omega(\nu+1)}(e, \langle X \cup \{x_1, \dots, x_i\}\rangle^0)\\ &= \hat{w}_{\omega(\nu+1)}(e, \langle Y \cup \{y_1, \dots, y_i\}\rangle^0),\end{aligned}$$

the latter equality following from (ii) and induction. By Theorem 5.2 there is $Y' \in \mathcal{C}'$ such that $Y \cup \{y_1, \dots, y_i\} \subseteq Y'$ and

$$\hat{w}_{\omega(\nu+1)}(e, \langle Y'\rangle^0) \geq \hat{w}_{\omega(\nu+1)}(e, \langle Y \cup \{y_1, \dots, y_i\}\rangle^0) + 1,$$

so there is a $y_{i+1} \in Y' \backslash (Y \cup \{y_1, \dots, y_i\})$ such that $[U(y_{i+1})] \sim_{\omega(\nu+1)} e$. Then $U(p^{n'_{i+1}}y_{i+1}) =_{\omega(\nu+1)} U(p^{m'_{i+1}}x_{i+1})$ for some $m'_{i+1}, n'_{i+1} \in \omega$. Now apply Lemma 4.4 to the module $\langle S, x_1, \dots, x_i, x_{i+1}\rangle^0$ with decomposition basis $X \cup \{x_1, \dots, x_{i+1}\}$ and submodule $\langle S, x_1, \dots, x_i\rangle$. This applies since

$$\begin{aligned}\langle S, x_1, \dots, x_i\rangle \cap \langle X \cup \{x_1, \dots, x_{i+1}\}\rangle &= \langle X \cup \{x_1, \dots, x_i\}\rangle\\ &= \langle\langle S, x_1, \dots, x_i\rangle \cap (X \cup \{x_1, \dots, x_{i+1}\})\rangle.\end{aligned}$$

We may similarly apply Lemma 4.4 to $\langle T, y_1, \dots, y_{i+1}\rangle^0$, $Y \cup \{y_1, \dots, y_{i+1}\}$ and $\langle T, y_1, \dots, y_i\rangle$. This gives us $k_{i+1} \in \omega$ such that for all $s \in \langle S, x_1, \dots, x_i\rangle$, $t \in \langle T, y_1, \dots, y_i\rangle$ and $r \in \mathbb{Z}_p$,

$$\begin{aligned}&|rp^{k_{i+1}}x_{i+1} + s| = \min\{|rp^{k_{i+1}}x_{i+1}|, |s|\} \text{ and}\\ &|rp^{k_{i+1}}y_{i+1} + t| = \min\{|rp^{k_{i+1}}y_{i+1}|, |t|\}.\end{aligned}$$

Now let $m_{i+1} = k_{i+1} + m'_{i+1}$, $n_{i+1} = k_{i+1} + n'_{i+1}$ and define f_{i+1} extending f_i by sending $p^{m_{i+1}} x_{i+1}$ to $p^{n_{i+1}} y_{i+1}$. By induction, f_i satisfies conditions (i)-(iii) of $I_{\nu+1}$ with associated X_i and Y_i, and it is easy to verify that f_{i+1} does as well with associated $X_{i+1} = X_i \cup \{p^{m_{i+1}} x_{i+1}\} \in \mathcal{C}$ and $Y_{i+1} = Y_i \cup \{p^{n_{i+1}} y_{i+1}\} \in \mathcal{C}'$. This completes the induction.

Let $x'_i = p^{m_i} x_i$ and $y'_i = p^{n_i} y_i$ $(i = 1, \ldots, m)$. Then $X_m = X \cup \{x'_1, \ldots, x'_m\}$, so there is a least n such that $p^{n+1} a \in \langle X_m \rangle$. Choose $k \in \omega$ sufficiently large so that for any given $j \leq n$ if every $x \in p^j a + \langle S, x'_1, \ldots, x'_m, p^{j+1} a \rangle$ has height $< \omega(\nu + 1)$, then every such x has height $< \omega\nu + k$, and if $length(tM) < \omega(\nu + 1)$ then $length(tM) < \omega\nu + k$. This is possible since the submodules involved are nice by Lemmas 5.4 and 4.5. The map

$$f_m : \langle S, x'_1, \ldots, x'_m \rangle \to \langle T, y'_1, \ldots, y'_m \rangle$$

preserves heights up to $\omega(\nu + 1)$ hence it preserves heights up to $\omega\nu + k + n + 1$. By Lemma 4.6, f_m extends to an $\omega\nu + k$-height-preserving isomorphism g with $domain(g) = \langle S, x'_1, \ldots, x'_m, a \rangle$. Thus g satisfies conditions (i) and (ii) on I_ν. Also since f_m satisfies condition (iii), so does g, since we may use the same X_m and Y_m because $a \in \langle X_m \rangle^0$, $X_m \subseteq domain(g)$ and $Y_m \subseteq range(g)$. Thus $g \in I_\nu$, as desired. By symmetry, condition (ii) of Theorem 3.1 is satisfied and we obtain $M \equiv_\delta N$. □

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Jacoby Consulting, Long Beach, California
E-mail address: `cjacoby@jacobyconsulting.com`

Department of Mathematics, Sacred Heart University, Fairfield, CT 06825, USA
E-mail address: `lothp@sacredheart.edu`