2015

Pure Injective and *-pure Injective LCA Groups

Peter Loth
Sacred Heart University, lothp@sacredheart.edu

REND. SEM. MAT. UNIV. PADOVA, Vol. 133 (2015)
DOI 10.4171/RSMUP/133-4

Pure injective and $*$-pure injective LCA groups

PETER LOTH

ABSTRACT - A proper short exact sequence $0 \to A \to B \to C \to 0$ in the category $\mathfrak{L}$ of locally compact abelian (LCA) groups is called $*$-*pure* if the induced sequence $0 \to A[n] \to B[n] \to C[n] \to 0$ is proper exact for all positive integers n. An LCA group is called $*$-*pure injective in* $\mathfrak{L}$ if it has the injective property relative to all $*$-pure sequences in $\mathfrak{L}$. In this paper, we give a complete description of the $*$-pure injectives in $\mathfrak{L}$. They coincide with the injectives in $\mathfrak{L}$ and therefore with the pure injectives in $\mathfrak{L}$. Dually, we determine the topologically pure projectives in $\mathfrak{L}$.

MATHEMATICS SUBJECT CLASSIFICATION (2010). 20K35, 22B05, 20K25, 20K40, 20K45.

KEYWORDS. Locally compact abelian groups, pure injectives, $*$-pure injectives, topologically pure projectives.

1. Introduction

All groups considered in this paper are Hausdorff abelian topological groups and they will be written additively. For a group G and a positive integer n, let $nG = \{nx : x \in G\}$ and $G[n] = \{x \in G : nx = 0\}$. Let $\mathfrak{L}$ denote the category of locally compact abelian groups with continuous homomorphisms as morphisms. In [15], Moskowitz developed a homological theory in the category $\mathfrak{L}$ and studied the functors Hom, $\otimes$, Tor and Ext on certain subcategories of $\mathfrak{L}$. Later Fulp and Griffith ([9], [10]) extended Moskowitz's construction of the functor Ext to the category $\mathfrak{L}$. Following Fulp and Griffith ([9]), we call a morphism *proper* if it is open onto its image. An exact sequence

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_n} G_n$$

(*) Indirizzo dell'A.: Department of Mathematics, Sacred Heart University, 5151 Park Avenue, Fairfield, Connecticut 06825, USA.
E-mail: lothp@sacredheart.edu

in $\mathfrak{L}$ is called *proper exact* if each morphism ϕ_i is proper. A proper short exact sequence $E: 0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ is called an *extension of* A *by* C (*in* $\mathfrak{L}$) and $\mathrm{Ext}(C, A)$ denotes the group of extensions of A by C (see [9]). Then the extension E is pure if and only if the induced sequence

$$E_n : 0 \to A[n] \to B[n] \to C[n] \to 0$$

is exact for all positive integers n (see [6, Theorem 29.1]). The elements represented by pure extensions of A by C form a subgroup of $\mathrm{Ext}(C, A)$ which is denoted by $\mathrm{Pext}(C, A)$. If each sequence E_n is proper exact, we call the extension E $*$-*pure*.

The concept of purity plays an important role in abelian group theory (see for instance [6]). In [7], Fulp studied pure extensions in the category $\mathfrak{L}$. As it was pointed out by Armacost [1], much of the paper is based on [7, Proposition 2] (stating that the dual of a pure extension is pure) which is unfortunately not valid for all groups in $\mathfrak{L}$.

In this paper, we continue our study of $*$-pure extensions started in [13] and give a complete description of the $*$-pure injectives in the category of locally compact abelian groups. Let $\mathfrak{C}$ denote the class of all groups X in $\mathfrak{L}$ such that X is connected or X is a torsion-free group which is either discrete or a topological torsion group (for the definition, see Section 2). Then a group G in $\mathfrak{L}$ has the property that every $*$-pure extension of G by a group in $\mathfrak{C}$ splits if and only if G has the form $R \oplus T$ where R is a vector group and T is a toral group (Theorem 3.7). Consequently, the $*$-pure injectives in $\mathfrak{L}$ coincide not only with the injectives in $\mathfrak{L}$ but also with the pure injectives in $\mathfrak{L}$ (see Theorem 4.1 and Corollary 4.2). Recall that a proper exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ is said to be *topologically pure* if for each positive integer n, the induced sequence

$$0 \to \overline{nA} \to \overline{nB} \to \overline{nC} \to 0$$

is proper exact (see [13]). Using Pontrjagin duality, we obtain the following result: A group in $\mathfrak{L}$ is topologically pure projective if and only if it has the form $R \oplus F$ where R is a vector group and F is a free group (see Corollary 4.3).

The group of real numbers with the usual topology is denoted by $\mathbb{R}$, $\mathbb{Z}$ is the group of integers, $\mathbb{Q}$ is the group of rationals taken discrete and $\mathbb{T}$ denotes the quotient $\mathbb{R}/\mathbb{Z}$. By $\mathbb{Z}(p^\infty)$ we mean the quasicyclic group and F_p is the additive group of the p-adic number field with the usual topology. For any groups G and H in $\mathfrak{L}$, let $\mathrm{Hom}(G, H)$ denote the group of all continuous homomorphisms from G to H. The identity component of G is given by G_0 and the union of all compact subgroups of G is denoted by $B(G)$. Notice that $B(G)$ is a closed subgroup of G (cf. [4, Proposition 3.3.6]).

The Pontrjagin dual of G is

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{T}),$$

endowed with the compact-open topology. All isomorphisms are understood to be topological isomorphisms and all considered direct sums are topological direct sums. We mostly follow the standard notation in [6] for abelian groups and [1] for locally compact abelian groups. For background information on abelian topological groups and Pontrjagin duality, we refer the reader to the books [4] and [12].

2. Preliminaries

A group G in $\mathfrak{L}$ is called *injective in* $\mathfrak{L}$ if for every proper exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ and every $\alpha \in \mathrm{Hom}(A, G)$ there is a $\beta \in \mathrm{Hom}(B, G)$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \alpha\downarrow & \swarrow\beta & & & & & \\ & & G & & & & & & \end{array}$$

is commutative. Dually, G is called *projective in* $\mathfrak{L}$ if for every proper exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ and every $\gamma \in \mathrm{Hom}(G, C)$ there is a $\delta \in \mathrm{Hom}(G, B)$ such that the diagram

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & \delta\swarrow & \downarrow\gamma & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

is commutative. Dixmier [3] and later Moskowitz [15] independently characterized the injectives in $\mathfrak{L}$:

THEOREM 2.1 ([3], [15]). *The following are equivalent for a group G in $\mathfrak{L}$:*

(1) *G is injective in $\mathfrak{L}$;*
(2) *$G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$ where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*

Using Pontrjagin duality, Moskowitz [15] proved the following:

THEOREM 2.2 ([15]). *The following are equivalent for a group G in $\mathfrak{L}$:*

(1) *G is projective in $\mathfrak{L}$;*
(2) *$G \cong \mathbb{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbb{Z}$ where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*

Using the notion of proper morphisms, Fulp and Griffith [9] developed the (discrete) group-valued extension functor Ext for the category $\mathfrak{L}$, generalizing both the functor Ext as defined in (discrete) abelian group theory and the functor Ext studied by Moskowitz [15]. We would like to point out the unfortunate fact that, at the same time, another use of the term "proper" exists; it is used by some authors in topology as a synonym for stably closed (what Engelking [5] calls "perfect"). The following basic properties will be useful:

Proposition 2.3 ([9]). *If G is a discrete group, then* $\operatorname{Ext}(\mathbb{T}, G) \cong G$. *Hence the range of* Ext *is all of the discrete groups.*

Theorem 2.4 ([9]). *Let G be a group in $\mathfrak{L}$. If $\{H_i : i \in I\}$ is a collection of groups in $\mathfrak{L}$ such that H_i is compact for almost all i, then* $\operatorname{Ext}\big(G, \prod_{i\in I} H_i\big) \cong \prod_{i\in I} \operatorname{Ext}(G, H_i)$.

In [9], Fulp and Griffith proved that the Hom-Ext sequences are exact except possibly at the right end. Then, in [10], they showed that Ext is right-exact; in fact, it was shown that $\operatorname{Ext}^n = 0$ for all $n \geq 2$.

Theorem 2.5 ([9], [10]). *Let G be a group in $\mathfrak{L}$ and let $0 \to A \to B \to C \to 0$ be a proper exact sequence in $\mathfrak{L}$. Then the following induced sequences are exact:*

$$(1)\quad 0 \to \operatorname{Hom}(G, A) \to \operatorname{Hom}(G, B) \to \operatorname{Hom}(G, C) \to \operatorname{Ext}(G, A) \to \operatorname{Ext}(G, B) \to \operatorname{Ext}(G, C) \to 0,$$

$$(2)\quad 0 \to \operatorname{Hom}(C, G) \to \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G) \to \operatorname{Ext}(C, G) \to \operatorname{Ext}(B, G) \to \operatorname{Ext}(A, G) \to 0.$$

Using the right-exactness of Ext, Fulp and Griffith were able to improve Theorem 2.1:

Theorem 2.6 ([10]). *The following are equivalent for a group G in $\mathfrak{L}$:*

(1) $G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$ *where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*
(2) $\operatorname{Ext}(C, G) = 0$ *for all connected groups C in $\mathfrak{L}$.*

A group G in $\mathfrak{L}$ is called a *topological torsion group* if $\lim_{n\to\infty} n!x = 0$ for all $x \in G$. Robertson [16] established several characterizations of topological torsion groups including the following:

THEOREM 2.7 ([16]). *A group G in $\mathfrak{L}$ is a topological torsion group if and only if both G and $\widehat{G}$ are totally disconnected.*

Now let $\{G_i : i \in I\}$ be a collection of groups in $\mathfrak{L}$ and let H_i be a compact open subgroup of G_i for every $i \in I$. Then the *local direct product of the groups G_i with respect to the subgroups H_i* is defined to be the group

$$G = \Big\{(x_i) \in \prod_{i\in I} G_i : x_i \in H_i \text{ for almost all } i\Big\}$$

and is topologized so that it contains $\prod_{i\in I} H_i$ (with its compact product topology) as an open subgroup (cf. [12, (6.16)]). The group G is in $\mathfrak{L}$ and is denoted by $LP_{i\in I}(G_i, H_i)$. Braconnier [2] and Vilenkin [18] proved independently that every topological torsion group G can be decomposed into a local direct product of its *p-components*

$$G_p = \{x \in G : \lim_{n\to\infty} p^n x = 0\}$$

belonging to different primes p:

THEOREM 2.8 ([2], [18]). *Let G be a topological torsion group and let H be any compact open subgroup of G. Then G_p is a closed subgroup of G for every prime p and G is isomorphic to the local direct product $LP_{p\in\mathbf{P}}(G_p, H_p)$.*

Let p be a prime and G a group in $\mathfrak{L}$. Then G is called a *topological p-group* if $G = G_p$. If G contains a dense divisible subgroup, then G is said to be *densely divisible* (see [16]). The next result will be needed:

PROPOSITION 2.9 ([1]). *Let G be a nontrivial topological p-group. If G is densely divisible, then G contains a closed subgroup D such that $D \cong F_p$ or $D \cong \mathbb{Z}(p^\infty)$.*

3. Splitting $*$-pure extensions

For groups A and C in $\mathfrak{L}$, let ${}^*\mathrm{Pext}(C,A)$ denote the set of elements $E \in \mathrm{Ext}(C,A)$ such that E is equivalent to some $*$-pure extension of A by C. Then ${}^*\mathrm{Pext}(C,A) \subseteq \mathrm{Pext}(C,A)$ and ${}^*\mathrm{Pext}(C,A) = 0$ if and only if every $*$-pure extension of A by C splits (cf. [13]).

LEMMA 3.1. *Let A and C be groups in $\mathfrak{L}$. Then:*

(1) *If C is torsion-free, then* ${}^{*}\mathrm{Pext}(C,A)=\mathrm{Ext}(C,A)$.
(2) *If $A=H\oplus K$ for some groups H and K in $\mathfrak{L}$ and* ${}^{*}\mathrm{Pext}(C,A)=0$, *then* ${}^{*}\mathrm{Pext}(C,H)=0$.

PROOF. (1) Suppose $E:0\to A\xrightarrow{\phi}B\to C\to 0$ is an extension in $\mathfrak{L}$ where C is torsion-free. Then E is pure and $C[n]=0$ for every positive integer n. Since ϕ is proper and injective, each map $\phi|_{A[n]}:A[n]\to B[n]$ is proper. It follows that each sequence $0\to A[n]\to B[n]\to C[n]\to 0$ is proper exact, hence E is $*$-pure.

(2) Let $E:0\to H\xrightarrow{\psi}B\to C\to 0$ be a $*$-pure sequence. Then

$$E':0\to H\oplus K\to B\oplus K\to C\to 0$$

is an extension in $\mathfrak{L}$ and $0\to H[n]\oplus K[n]\to B[n]\oplus K[n]\to C[n]\to 0$ is proper exact for all positive integers n. If the sequence E' splits, then $\psi(H)\oplus K$ is a direct summand of $B\oplus K$. But then $\psi(H)$ is a direct summand of B (see the proof of [1, Lemma 9.11]), hence the sequence E splits. □

The proof of [13, Theorem 4.3(1)] shows the following:

PROPOSITION 3.2. *If a group G in $\mathfrak{L}$ satisfies* ${}^{*}\mathrm{Pext}(C,G)=0$ *for all connected groups C in $\mathfrak{L}$, then there is a closed subgroup H of G such that $G=G_0\oplus H$ and $G_0\cong\mathbb{R}^n\times\mathbb{T}^{\mathfrak{m}}$ for some nonnegative integer n and cardinal $\mathfrak{m}$.*

Let $E:0\to A\xrightarrow{\phi}B\to C\to 0$ be a proper exact sequence in $\mathfrak{L}$ and $\alpha\in\mathrm{Hom}(A,G)$ where G is a group in $\mathfrak{L}$. Then there is a standard pushout diagram for α and ϕ

$$\begin{array}{rccccccccc} E: & 0 & \to & A & \xrightarrow{\phi} & B & \xrightarrow{\pi} & C & \to & 0\\ & & & \alpha\downarrow & & \downarrow & & \| & & \\ \alpha E: & 0 & \to & G & \xrightarrow{\phi'} & X & \xrightarrow{\pi'} & C & \to & 0 \end{array}$$

(cf. [9, Proposition 2.5]). Recall that $X=(G\oplus B)/N$ where $N=\{(-\alpha(a),\phi(a)):a\in A\}$ is a closed subgroup of $G\oplus B$, $\phi':g\mapsto(g,0)+N$ and $\pi':(g,b)+N\mapsto\pi(b)$. Further, αE is a proper exact sequence in $\mathfrak{L}$ (see [9, p. 350]). The next result will be useful:

LEMMA 3.3. *Let* $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\pi} C \to 0$ *be a proper exact sequence in* $\mathfrak{L}$ *such that* A *is divisible and* $B[n]$ *is* σ*-compact for all positive integers* n*. Suppose that* G *is a group in* $\mathfrak{L}$ *and* $\alpha \in \mathrm{Hom}(A, G)$*. Then both* E *and* αE *are* $*$*-pure.*

PROOF. Let n be a positive integer. The exact sequence E is pure because A is divisible, therefore the induced sequence

$$E_n : 0 \to A[n] \xrightarrow{\phi|_{A[n]}} B[n] \xrightarrow{\pi|_{B[n]}} C[n] \to 0$$

is exact. The map $\phi|_{A[n]}$ is proper and since $B[n]$ is σ-compact, $\pi|_{B[n]}$ is proper by the open mapping theorem (see [12, (5.29)]), hence E_n is proper exact. Therefore, E is $*$-pure. The maps α and ϕ have a standard pushout diagram

$$\begin{array}{ccccccccccc} E: & 0 & \to & A & \xrightarrow{\phi} & B & \xrightarrow{\pi} & C & \to & 0 \\ & & & \alpha\downarrow & & \downarrow & & \| & & \\ \alpha E: & 0 & \to & G & \xrightarrow{\phi'} & X & \xrightarrow{\pi'} & C & \to & 0 \end{array}$$

and αE is proper exact. Since E is pure, the sequence αE is pure ([11, Lemma 26]), hence the induced sequence $(\alpha E)_n : 0 \to G[n] \to X[n] \to C[n] \to 0$ is exact. We need to show that $(\alpha E)_n$ is proper exact. Notice that the continuous surjective homomorphism $\varphi = \pi'|_{X[n]} : X[n] \to C[n]$ is open if and only if the induced map $\overline{\varphi} : X[n]/\ker\varphi \to C[n]$ is an isomorphism in $\mathfrak{L}$ (cf. [12, p. 41]). The group $N = \{(-\alpha(a), \phi(a)) : a \in A\}$ is divisible and therefore pure in $G \oplus B$, hence $X[n] = ((G \oplus B)/N)[n] = (G[n] \oplus B[n] + N)/N$. Notice that both $G[n] \oplus B[n] + N$ and $G[n] \oplus 0 + N$ are locally compact since $X[n]$ and $\ker\varphi = (G[n] \oplus 0 + N)/N$ are locally compact ([12, (5.25)]). The group $X[n]/\ker\varphi$ is equal to

$$\frac{(G[n] \oplus B[n] + N)/N}{(G[n] \oplus 0 + N)/N} \cong \frac{G[n] \oplus B[n] + N}{G[n] \oplus 0 + N} = \frac{(0 \oplus B[n]) + (G[n] \oplus 0 + N)}{G[n] \oplus 0 + N}$$

(cf. [12, (5.35)]) and by the second isomorphism theorem in $\mathfrak{L}$ (see [9, Theorem 3.3]), the latter group is isomorphic to

$$\frac{0 \oplus B[n]}{(0 \oplus B[n]) \cap (G[n] \oplus 0 + N)} = \frac{0 \oplus B[n]}{0 \oplus \phi(A[n])} \cong C[n]$$

since $B[n]$ is σ-compact. Thus we have an isomorphism from $X[n]/\ker\varphi$ to $C[n]$ given by $((g, b) + N) + \ker\varphi \mapsto \pi(b)$ $(g \in G[n], b \in B[n])$. Since this map coincides with $\overline{\varphi}$ it follows that $\varphi : X[n] \to C[n]$ is open. Therefore, αE is $*$-pure. □

PROPOSITION 3.4. *Let G be a totally disconnected group in $\mathfrak{L}$ such that* $\mathrm{Ext}(\widehat{\mathbb{Q}}, G) = 0$. *Then G is a topological torsion group.*

PROOF. By Theorem 2.7, it suffices to show that $\widehat{G}$ is totally disconnected. To prove this, we argue as in the proof of [14, Theorem 2.7 (ii) $\Rightarrow$ (iii)]. First, notice that the quotient $G/B(G)$ is discrete (cf. [12, (9.26)(a)]) and torsion-free, and that $(\mathbb{Q}/\mathbb{Z})\widehat{\ }$ is compact since $\mathbb{Q}/\mathbb{Z}$ is discrete ([12, (23.17)]). The proper exact sequence $0 \to B(G) \to G \to G/B(G) \to 0$ gives rise to the exact sequence $0 = \mathrm{Ext}(\widehat{\mathbb{Q}}, G) \to \mathrm{Ext}(\widehat{\mathbb{Q}}, G/B(G)) \to 0$. But then exactness of the sequence

$$0 = \mathrm{Hom}((\mathbb{Q}/\mathbb{Z})\widehat{\ }, G/B(G)) \to \mathrm{Ext}(\widehat{\mathbb{Z}}, G/B(G)) \to \mathrm{Ext}(\widehat{\mathbb{Q}}, G/B(G)) = 0$$

yields $G/B(G) \cong \mathrm{Ext}(\widehat{\mathbb{Z}}, G/B(G)) = 0$ by Proposition 2.3, thus G coincides with $B(G)$. Since $\widehat{G}_0$ is the annihilator of $B(G)$ in $\widehat{G}$ (cf. [12, (24.17)]), it follows that $\widehat{G}$ is totally disconnected. □

The following lemma will be needed:

LEMMA 3.5 [14, Lemma 2.6]. *Suppose that G is a group in $\mathfrak{L}$ possessing a compact open subgroup. Then G is densely divisible if and only if G/C is divisible for every compact open subgroup C of G.*

PROPOSITION 3.6. *Suppose that G is a topological torsion group such that* $\mathrm{Ext}(X, G) = 0$ *for every torsion-free group X in $\mathfrak{L}$ which is either discrete or a topological torsion group. Then G is densely divisible.*

PROOF. Our proof is similar to the second part of the proof of [8, Theorem 7]. Let C be a compact open subgroup of G and set $A = G/C$. Then for any torsion-free group X in $\mathfrak{L}$ which is discrete or a topological torsion group, exactness of the sequence

$$0 = \mathrm{Ext}(X, G) \to \mathrm{Ext}(X, A) \to 0$$

yields $\mathrm{Ext}(X, A) = 0$. Recall that a discrete group H is said to be *cotorsion* if $\mathrm{Ext}(J, H) = 0$ for every discrete torsion-free group J (see [6, page 232]). Then the group A is cotorsion. Since A is also torsion, we have $A = B \oplus D$ for some bounded group B and divisible group D (see [6, Corollary 54.4]). A bounded group is a direct sum of cyclic groups ([6, Theorem 17.2]), so if $B \neq 0$, then B contains a direct summand $B' \cong \mathbb{Z}/p^n\mathbb{Z}$ for some prime p and positive integer n. By [13, Example 2.4], there is a non-splitting proper exact sequence

$$0 \to B' \to K \to L \to 0$$

in $\mathfrak{L}$ where L is torsion-free and $\widehat{L}$ is a p-group, hence $\widehat{L}$ is a topological torsion group. By Theorem 2.7, L is a topological torsion group and we have $\mathrm{Ext}(L, A) = 0$. But then Theorem 2.4 shows that $\mathrm{Ext}(L, B') = 0$ which is impossible. Therefore $B = 0$ and it follows from Lemma 3.5 that G is densely divisible. □

Let $\mathfrak{C}$ denote the class of groups X in $\mathfrak{L}$ such that X is connected or X is a torsion-free group which is either discrete or a topological torsion group. Then the groups G in $\mathfrak{L}$ having the property that every $*$-pure extension of G by a group in $\mathfrak{C}$ splits can be characterized as follows:

THEOREM 3.7. *A group G in $\mathfrak{L}$ satisfies* ${}^{*}\mathrm{Pext}(X, G) = 0$ *for all groups X in $\mathfrak{C}$ if and only if $G \cong \mathbb{R}^n \times \mathbb{T}^{\mathfrak{m}}$ for some nonnegative integer n and cardinal $\mathfrak{m}$.*

PROOF. Sufficiency follows from Theorem 2.1. Conversely, suppose ${}^{*}\mathrm{Pext}(X, G) = 0$ for all groups X in $\mathfrak{C}$. By Proposition 3.2, we have $G = G_0 \oplus H$ where $G_0 \cong \mathbb{R}^n \times \mathbb{T}^{\mathfrak{m}}$ for some nonnegative integer n and cardinal $\mathfrak{m}$. Due to Lemma 3.1(ii), ${}^{*}\mathrm{Pext}(X, H) = 0$ for all groups X in $\mathfrak{C}$. Then by Lemma 3.1(i), Proposition 3.4 and Proposition 3.6, H is a densely divisible topological torsion group. By Theorem 2.8, H can be identified with a local direct product of its p-components

$$H_p = \{x \in H : \lim_{n \to \infty} p^n x = 0\}$$

belonging to different primes p. Assume $H \neq 0$. Then there exists a prime p such that $H_p \neq 0$. Since the projection map $H \to H_p$ is continuous, H_p is densely divisible, so by Proposition 2.9 it contains a closed subgroup D such that $D \cong F_p$ or $D \cong \mathbb{Z}(p^\infty)$. In either case, D is a divisible σ-compact group in $\mathfrak{L}$ ([12, (10.5)]). For the inclusion map $\alpha : D \to H$ and a connected group X in $\mathfrak{L}$, consider the exact sequence

$$0 = \mathrm{Hom}(X, H/D) \to \mathrm{Ext}(X, D) \xrightarrow{\alpha_*} \mathrm{Ext}(X, H).$$

To show that $\mathrm{Ext}(X, D) = 0$, let $E : 0 \to D \xrightarrow{\phi} F \to X \to 0 \in \mathrm{Ext}(X, D)$. The group $F/\phi(D) \cong X$ is σ-compact since it is connected ([12, (9.14)]) and $\phi(D)$ is σ-compact, hence F is σ-compact ([17, Theorem 6.10(c)]) and it follows that every group $F[n]$ is σ-compact. By Lemma 3.3, $\alpha_*(E) = \alpha E$

is a $*$-pure extension, so it splits. Since α_* is injective, E splits as well and we obtain $\mathrm{Ext}(X, D) = 0$. But then Theorem 2.6 shows that D is connected, a contradiction. Consequently, $H = 0$ and we have $G \cong \mathbb{R}^n \times \mathbb{T}^{\mathfrak{m}}$, as desired. □

4. Injective and projective properties

A group G in $\mathfrak{L}$ is called *pure injective in* $\mathfrak{L}$ if it has the injective property relative to all pure extensions in $\mathfrak{L}$, that is, if for every pure proper exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ and every $\alpha \in \mathrm{Hom}(A, G)$ there is a $\beta \in \mathrm{Hom}(B, G)$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \alpha\downarrow & \swarrow\beta & & & & & \\ & & G & & & & & & \end{array}$$

is commutative. Similarly, we call a group in $\mathfrak{L}$ *$*$-pure injective in* $\mathfrak{L}$ if it has the injective property relative to all $*$-pure extensions. Then we have:

THEOREM 4.1. *The following are equivalent for a group G in $\mathfrak{L}$:*

(1) *G is $*$-pure injective in $\mathfrak{L}$;*
(2) $^*\mathrm{Pext}(X, G) = 0$ *for all groups X in $\mathfrak{L}$;*
(3) *$G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$ where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*

PROOF. Suppose that G is $*$-pure injective in $\mathfrak{L}$. Then every $*$-pure extension $0 \to G \to B \to X \to 0$ splits because there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & G & \to & B & \to & X & \to & 0 \\ & & \| & \swarrow & & & & & \\ & & G. & & & & & & \end{array}$$

Consequently, (1) implies (2). By Theorem 3.7, (2) implies (3). The groups of the form $\mathbb{R}^n \times \mathbb{T}^{\mathfrak{m}}$ are injective in $\mathfrak{L}$ (Theorem 2.1), hence (3) implies (1). □

By the theorem above, the $*$-pure injectives in $\mathfrak{L}$ are exactly the injectives in $\mathfrak{L}$. As an immediate consequence, we obtain a complete description of the pure injectives in $\mathfrak{L}$. This extends [14, Theorem 2.7] and shows that the result on discrete and compact injectives in $\mathfrak{L}$ as stated in [7, Proposition 8] is incorrect.

COROLLARY 4.2. *The following are equivalent for a group G in $\mathfrak{L}$:*

(1) *G is pure injective in $\mathfrak{L}$;*
(2) $\mathrm{Pext}(X, G) = 0$ *for all groups X in $\mathfrak{L}$;*
(3) *$G \cong \mathbb{R}^n \oplus \mathbb{T}^{\mathfrak{m}}$ where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*

Recall that a proper exact sequence $E : 0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ is said to be *topologically pure* if the induced sequence

$$0 \to \overline{nA} \to \overline{nB} \to \overline{nC} \to 0$$

is proper exact for all positive integers n (see [13]). Pontrjagin duality shows that the sequence E is topologically pure if and only if its dual sequence

$$0 \to \widehat{C} \to \widehat{B} \to \widehat{A} \to 0$$

is $*$-pure (see [13, Corollary 2.6]). We call a group G in $\mathfrak{L}$ *topologically pure projective in* $\mathfrak{L}$ if it has the projective property relative to all topologically pure extensions, in other words, if for every topologically pure exact sequence $0 \to A \to B \to C \to 0$ and every $\gamma \in \mathrm{Hom}(G, C)$ there is a $\delta \in \mathrm{Hom}(G, B)$ such that the diagram

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & \delta \swarrow & \downarrow \gamma & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

is commutative. Then dualization of Theorem 4.1 yields the following result which extends [13, Theorem 4.4(3)]:

COROLLARY 4.3. *The following are equivalent for a group G in $\mathfrak{L}$:*

(1) *G is topologically pure projective in $\mathfrak{L}$;*
(2) *every topologically pure sequence $0 \to A \to B \to G \to 0$ splits;*
(3) *$G \cong \mathbb{R}^n \oplus \bigoplus_{\mathfrak{m}} \mathbb{Z}$ where n is a nonnegative integer and $\mathfrak{m}$ is a cardinal.*

Acknowledgments. The author would like to thank the referee for the valuable comments and suggestions which helped to improve this paper.

REFERENCES

[1] D. L. ARMACOST, *The Structure of Locally Compact Abelian Groups*, Marcel Dekker Inc., New York, 1981.
[2] J. BRACONNIER, *Sur les groupes topologiques localement compacts*, J. Math. Pures Appl., N.S. **27** (1948), pp. 1–85.

[3] J. Dixmier, *Quelques propriétés des groupes abéliens localement compacts*, Bull. Sci. Math. (2) **81** (1957), pp. 38–48.

[4] D. N. Dikranjan, I. R. Prodanov and L. N. Stoyanov, *Topological Groups: Dualities and Minimal Group Topologies*, Marcel Dekker Inc., New York, Basel, 1989.

[5] R. D. Engelking, *General Topology*, revised and completed edition, Heldermann Verlag, Berlin, 1989.

[6] L. Fuchs, *Infinite Abelian Groups*, Vol. **I**, Academic Press, New York, 1970.

[7] R. O. Fulp, *Homological study of purity in locally compact groups*, Proc. London Math. Soc. **21** (1970), pp. 501–512.

[8] R. O. Fulp, *Splitting locally compact abelian groups*, Michigan Math. J. **19** (1972), pp. 47–55.

[9] R. O. Fulp and P. Griffith, *Extensions of locally compact abelian groups I*, Trans. Amer. Math. Soc. **154** (1971), pp. 341–356.

[10] R. O. Fulp and P. Griffith, *Extensions of locally compact abelian groups II*, Trans. Amer. Math. Soc. **154** (1971), pp. 357–363.

[11] P. A. Griffith, *Infinite Abelian Group Theory*, The University of Chicago Press, Chicago and London (1970).

[12] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. **I**, second edition, Springer Verlag, Berlin, 1979.

[13] P. Loth, *Topologically pure extensions*, in: Abelian Groups, Rings and Modules, Proceedings of the AGRAM 2000 Conference in Perth, Western Australia, July 9-15, 2000, Contemp. Math. **273**, Amer. Math. Soc. (2001), pp. 191–201.

[14] P. Loth, *Pure extensions of locally compact abelian groups*, Rend. Sem. Mat. Univ. Padova **116** (2006), pp. 31–40.

[15] M. Moskowitz, *Homological algebra in locally compact abelian groups*, Trans. Amer. Math. Soc. **127** (1967), pp. 361–404.

[16] L. C. Robertson, *Connectivity, divisibility and torsion*, Trans. Amer. Math. Soc. **128** (1967), pp. 482–505.

[17] M. Stroppel, *Locally Compact Groups*, EMS Textbooks in Mathematics, Eur. Math. Soc., Zürich, 2006.

[18] N. Vilenkin, *Direct decompositions of topological groups I*, Mat. Sb., N. S. **19** (61) (1946), pp. 85–154. [English translation from the Russian by E. Hewitt in A.M.S. Translations, Series 1, Volume 23, Providence, Rhode Island (1950).]

Manoscritto pervenuto in redazione il 9 agosto 2013.