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Pure injective and $*$ -pure injective LCA groups

PETER LOTH

ABSTRACT - A proper short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in the category \mathfrak{L} of locally compact abelian (LCA) groups is called $*$ -pure if the induced sequence $0 \rightarrow A[n] \rightarrow B[n] \rightarrow C[n] \rightarrow 0$ is proper exact for all positive integers n . An LCA group is called $*$ -pure injective in \mathfrak{L} if it has the injective property relative to all $*$ -pure sequences in \mathfrak{L} . In this paper, we give a complete description of the $*$ -pure injectives in \mathfrak{L} . They coincide with the injectives in \mathfrak{L} and therefore with the pure injectives in \mathfrak{L} . Dually, we determine the topologically pure projectives in \mathfrak{L} .

MATHEMATICS SUBJECT CLASSIFICATION (2010). 20K35, 22B05, 20K25, 20K40, 20K45.

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1. Introduction

All groups considered in this paper are Hausdorff abelian topological groups and they will be written additively. For a group G and a positive integer n , let $nG = \{nx : x \in G\}$ and $G[n] = \{x \in G : nx = 0\}$. Let \mathfrak{L} denote the category of locally compact abelian groups with continuous homomorphisms as morphisms. In [15], Moskowitz developed a homological theory in the category \mathfrak{L} and studied the functors Hom , \otimes , Tor and Ext on certain subcategories of \mathfrak{L} . Later Fulp and Griffith ([9], [10]) extended Moskowitz's construction of the functor Ext to the category \mathfrak{L} . Following Fulp and Griffith ([9]), we call a morphism *proper* if it is open onto its image. An exact sequence

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_n} G_n$$

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in \mathfrak{L} is called *proper exact* if each morphism ϕ_i is proper. A proper short exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathfrak{L} is called an *extension of A by C* (in \mathfrak{L}) and $\text{Ext}(C, A)$ denotes the group of extensions of A by C (see [9]). Then the extension E is pure if and only if the induced sequence

$$E_n : 0 \rightarrow A[n] \rightarrow B[n] \rightarrow C[n] \rightarrow 0$$

is exact for all positive integers n (see [6, Theorem 29.1]). The elements represented by pure extensions of A by C form a subgroup of $\text{Ext}(C, A)$ which is denoted by $\text{Pext}(C, A)$. If each sequence E_n is proper exact, we call the extension E **-pure*.

The concept of purity plays an important role in abelian group theory (see for instance [6]). In [7], Fulp studied pure extensions in the category \mathfrak{L} . As it was pointed out by Armacost [1], much of the paper is based on [7, Proposition 2] (stating that the dual of a pure extension is pure) which is unfortunately not valid for all groups in \mathfrak{L} .

In this paper, we continue our study of *-pure extensions started in [13] and give a complete description of the *-pure injectives in the category of locally compact abelian groups. Let \mathfrak{C} denote the class of all groups X in \mathfrak{L} such that X is connected or X is a torsion-free group which is either discrete or a topological torsion group (for the definition, see Section 2). Then a group G in \mathfrak{L} has the property that every *-pure extension of G by a group in \mathfrak{C} splits if and only if G has the form $R \oplus T$ where R is a vector group and T is a toral group (Theorem 3.7). Consequently, the *-pure injectives in \mathfrak{L} coincide not only with the injectives in \mathfrak{L} but also with the pure injectives in \mathfrak{L} (see Theorem 4.1 and Corollary 4.2). Recall that a proper exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathfrak{L} is said to be *topologically pure* if for each positive integer n , the induced sequence

$$0 \rightarrow \overline{nA} \rightarrow \overline{nB} \rightarrow \overline{nC} \rightarrow 0$$

is proper exact (see [13]). Using Pontrjagin duality, we obtain the following result: A group in \mathfrak{L} is topologically pure projective if and only if it has the form $R \oplus F$ where R is a vector group and F is a free group (see Corollary 4.3).

The group of real numbers with the usual topology is denoted by \mathbb{R} , \mathbb{Z} is the group of integers, \mathbb{Q} is the group of rationals taken discrete and \mathbb{T} denotes the quotient \mathbb{R}/\mathbb{Z} . By $\mathbb{Z}(p^\infty)$ we mean the quasicyclic group and F_p is the additive group of the p -adic number field with the usual topology. For any groups G and H in \mathfrak{L} , let $\text{Hom}(G, H)$ denote the group of all continuous homomorphisms from G to H . The identity component of G is given by G_0 and the union of all compact subgroups of G is denoted by $B(G)$. Notice that $B(G)$ is a closed subgroup of G (cf. [4, Proposition 3.3.6]).

The Pontrjagin dual of G is

$$\widehat{G} = \text{Hom}(G, \mathbb{T}),$$

endowed with the compact-open topology. All isomorphisms are understood to be topological isomorphisms and all considered direct sums are topological direct sums. We mostly follow the standard notation in [6] for abelian groups and [1] for locally compact abelian groups. For background information on abelian topological groups and Pontrjagin duality, we refer the reader to the books [4] and [12].

2. Preliminaries

A group G in \mathfrak{L} is called *injective in \mathfrak{L}* if for every proper exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathfrak{L} and every $\alpha \in \text{Hom}(A, G)$ there is a $\beta \in \text{Hom}(B, G)$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\ & & & & \alpha \downarrow & \nearrow \beta & & & \\ & & & & G & & & & \end{array}$$

is commutative. Dually, G is called *projective in \mathfrak{L}* if for every proper exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathfrak{L} and every $\gamma \in \text{Hom}(G, C)$ there is a $\delta \in \text{Hom}(G, B)$ such that the diagram

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \delta \nearrow & \downarrow \gamma & \\ 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \end{array}$$

is commutative. Dixmier [3] and later Moskowitz [15] independently characterized the injectives in \mathfrak{L} :

THEOREM 2.1 ([3], [15]). *The following are equivalent for a group G in \mathfrak{L} :*

- (1) G is injective in \mathfrak{L} ;
- (2) $G \cong \mathbb{R}^n \oplus \mathbb{T}^m$ where n is a nonnegative integer and m is a cardinal.

Using Pontrjagin duality, Moskowitz [15] proved the following:

THEOREM 2.2 ([15]). *The following are equivalent for a group G in \mathfrak{L} :*

- (1) G is projective in \mathfrak{L} ;
- (2) $G \cong \mathbb{R}^n \oplus \bigoplus_m \mathbb{Z}$ where n is a nonnegative integer and m is a cardinal.

Using the notion of proper morphisms, Fulp and Griffith [9] developed the (discrete) group-valued extension functor Ext for the category \mathfrak{L} , generalizing both the functor Ext as defined in (discrete) abelian group theory and the functor Ext studied by Moskowitz [15]. We would like to point out the unfortunate fact that, at the same time, another use of the term “proper” exists; it is used by some authors in topology as a synonym for stably closed (what Engelking [5] calls “perfect”). The following basic properties will be useful:

PROPOSITION 2.3 ([9]). *If G is a discrete group, then $\text{Ext}(\mathbb{T}, G) \cong G$. Hence the range of Ext is all of the discrete groups.*

THEOREM 2.4 ([9]). *Let G be a group in \mathfrak{L} . If $\{H_i : i \in I\}$ is a collection of groups in \mathfrak{L} such that H_i is compact for almost all i , then $\text{Ext}(G, \prod_{i \in I} H_i) \cong \prod_{i \in I} \text{Ext}(G, H_i)$.*

In [9], Fulp and Griffith proved that the Hom-Ext sequences are exact except possibly at the right end. Then, in [10], they showed that Ext is right-exact; in fact, it was shown that $\text{Ext}^n = 0$ for all $n \geq 2$.

THEOREM 2.5 ([9], [10]). *Let G be a group in \mathfrak{L} and let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a proper exact sequence in \mathfrak{L} . Then the following induced sequences are exact:*

- (1) $0 \rightarrow \text{Hom}(G, A) \rightarrow \text{Hom}(G, B) \rightarrow \text{Hom}(G, C) \rightarrow \text{Ext}(G, A) \rightarrow \text{Ext}(G, B) \rightarrow \text{Ext}(G, C) \rightarrow 0$,
- (2) $0 \rightarrow \text{Hom}(C, G) \rightarrow \text{Hom}(B, G) \rightarrow \text{Hom}(A, G) \rightarrow \text{Ext}(C, G) \rightarrow \text{Ext}(B, G) \rightarrow \text{Ext}(A, G) \rightarrow 0$.

Using the right-exactness of Ext , Fulp and Griffith were able to improve Theorem 2.1:

THEOREM 2.6 ([10]). *The following are equivalent for a group G in \mathfrak{L} :*

- (1) $G \cong \mathbb{R}^n \oplus \mathbb{T}^m$ where n is a nonnegative integer and m is a cardinal.
- (2) $\text{Ext}(C, G) = 0$ for all connected groups C in \mathfrak{L} .

A group G in \mathfrak{L} is called a *topological torsion group* if $\lim_{n \rightarrow \infty} n!x = 0$ for all $x \in G$. Robertson [16] established several characterizations of topological torsion groups including the following:

THEOREM 2.7 ([16]). *A group G in \mathfrak{L} is a topological torsion group if and only if both G and \widehat{G} are totally disconnected.*

Now let $\{G_i : i \in I\}$ be a collection of groups in \mathfrak{L} and let H_i be a compact open subgroup of G_i for every $i \in I$. Then the *local direct product of the groups G_i with respect to the subgroups H_i* is defined to be the group

$$G = \left\{ (x_i) \in \prod_{i \in I} G_i : x_i \in H_i \text{ for almost all } i \right\}$$

and is topologized so that it contains $\prod_{i \in I} H_i$ (with its compact product topology) as an open subgroup (cf. [12, (6.16)]). The group G is in \mathfrak{L} and is denoted by $LP_{i \in I}(G_i, H_i)$. Braconnier [2] and Vilenkin [18] proved independently that every topological torsion group G can be decomposed into a local direct product of its *p-components*

$$G_p = \{x \in G : \lim_{n \rightarrow \infty} p^n x = 0\}$$

belonging to different primes p :

THEOREM 2.8 ([2], [18]). *Let G be a topological torsion group and let H be any compact open subgroup of G . Then G_p is a closed subgroup of G for every prime p and G is isomorphic to the local direct product $LP_{p \in \mathbb{P}}(G_p, H_p)$.*

Let p be a prime and G a group in \mathfrak{L} . Then G is called a *topological p-group* if $G = G_p$. If G contains a dense divisible subgroup, then G is said to be *densely divisible* (see [16]). The next result will be needed:

PROPOSITION 2.9 ([1]). *Let G be a nontrivial topological p-group. If G is densely divisible, then G contains a closed subgroup D such that $D \cong F_p$ or $D \cong \mathbb{Z}(p^\infty)$.*

3. Splitting *-pure extensions

For groups A and C in \mathfrak{L} , let ${}^*\text{Pext}(C, A)$ denote the set of elements $E \in \text{Ext}(C, A)$ such that E is equivalent to some *-pure extension of A by C . Then ${}^*\text{Pext}(C, A) \subseteq \text{Pext}(C, A)$ and ${}^*\text{Pext}(C, A) = 0$ if and only if every *-pure extension of A by C splits (cf. [13]).

LEMMA 3.1. *Let A and C be groups in \mathfrak{Q} . Then:*

- (1) *If C is torsion-free, then ${}^*\text{Pext}(C, A) = \text{Ext}(C, A)$.*
- (2) *If $A = H \oplus K$ for some groups H and K in \mathfrak{Q} and ${}^*\text{Pext}(C, A) = 0$, then ${}^*\text{Pext}(C, H) = 0$.*

PROOF. (1) Suppose $E : 0 \rightarrow A \xrightarrow{\phi} B \rightarrow C \rightarrow 0$ is an extension in \mathfrak{Q} where C is torsion-free. Then E is pure and $C[n] = 0$ for every positive integer n . Since ϕ is proper and injective, each map $\phi|_{A[n]} : A[n] \rightarrow B[n]$ is proper. It follows that each sequence $0 \rightarrow A[n] \rightarrow B[n] \rightarrow C[n] \rightarrow 0$ is proper exact, hence E is $*$ -pure.

(2) Let $E : 0 \rightarrow H \xrightarrow{\psi} B \rightarrow C \rightarrow 0$ be a $*$ -pure sequence. Then

$$E' : 0 \rightarrow H \oplus K \rightarrow B \oplus K \rightarrow C \rightarrow 0$$

is an extension in \mathfrak{Q} and $0 \rightarrow H[n] \oplus K[n] \rightarrow B[n] \oplus K[n] \rightarrow C[n] \rightarrow 0$ is proper exact for all positive integers n . If the sequence E' splits, then $\psi(H) \oplus K$ is a direct summand of $B \oplus K$. But then $\psi(H)$ is a direct summand of B (see the proof of [1, Lemma 9.11]), hence the sequence E splits. \square

The proof of [13, Theorem 4.3(1)] shows the following:

PROPOSITION 3.2. *If a group G in \mathfrak{Q} satisfies ${}^*\text{Pext}(C, G) = 0$ for all connected groups C in \mathfrak{Q} , then there is a closed subgroup H of G such that $G = G_0 \oplus H$ and $G_0 \cong \mathbb{R}^n \times \mathbb{T}^m$ for some nonnegative integer n and cardinal m .*

Let $E : 0 \rightarrow A \xrightarrow{\phi} B \rightarrow C \rightarrow 0$ be a proper exact sequence in \mathfrak{Q} and $\alpha \in \text{Hom}(A, G)$ where G is a group in \mathfrak{Q} . Then there is a standard pushout diagram for α and ϕ

$$\begin{array}{ccccccccc} E : & 0 & \rightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\pi} & C & \rightarrow & 0 \\ & & & \alpha \downarrow & & \downarrow & & \parallel & & \\ \alpha E : & 0 & \rightarrow & G & \xrightarrow{\phi'} & X & \xrightarrow{\pi'} & C & \rightarrow & 0 \end{array}$$

(cf. [9, Proposition 2.5]). Recall that $X = (G \oplus B)/N$ where $N = \{(-\alpha(a), \phi(a)) : a \in A\}$ is a closed subgroup of $G \oplus B$, $\phi' : g \mapsto (g, 0) + N$ and $\pi' : (g, b) + N \mapsto \pi(b)$. Further, αE is a proper exact sequence in \mathfrak{Q} (see [9, p. 350]). The next result will be useful:

LEMMA 3.3. *Let $E : 0 \rightarrow A \xrightarrow{\phi} B \xrightarrow{\pi} C \rightarrow 0$ be a proper exact sequence in \mathfrak{L} such that A is divisible and $B[n]$ is σ -compact for all positive integers n . Suppose that G is a group in \mathfrak{L} and $\alpha \in \text{Hom}(A, G)$. Then both E and αE are $*$ -pure.*

PROOF. Let n be a positive integer. The exact sequence E is pure because A is divisible, therefore the induced sequence

$$E_n : 0 \rightarrow A[n] \xrightarrow{\phi|_{A[n]}} B[n] \xrightarrow{\pi|_{B[n]}} C[n] \rightarrow 0$$

is exact. The map $\phi|_{A[n]}$ is proper and since $B[n]$ is σ -compact, $\pi|_{B[n]}$ is proper by the open mapping theorem (see [12, (5.29)]), hence E_n is proper exact. Therefore, E is $*$ -pure. The maps α and ϕ have a standard pushout diagram

$$\begin{array}{ccccccccc} E : & 0 & \rightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\pi} & C & \rightarrow & 0 \\ & & & \alpha \downarrow & & \downarrow & & \parallel & & \\ \alpha E : & 0 & \rightarrow & G & \xrightarrow{\phi'} & X & \xrightarrow{\pi'} & C & \rightarrow & 0 \end{array}$$

and αE is proper exact. Since E is pure, the sequence αE is pure ([11, Lemma 26]), hence the induced sequence $(\alpha E)_n : 0 \rightarrow G[n] \rightarrow X[n] \rightarrow C[n] \rightarrow 0$ is exact. We need to show that $(\alpha E)_n$ is proper exact. Notice that the continuous surjective homomorphism $\varphi = \pi'|_{X[n]} : X[n] \rightarrow C[n]$ is open if and only if the induced map $\bar{\varphi} : X[n]/\ker \varphi \rightarrow C[n]$ is an isomorphism in \mathfrak{L} (cf. [12, p. 41]). The group $N = \{(-\alpha(a), \phi(a)) : a \in A\}$ is divisible and therefore pure in $G \oplus B$, hence $X[n] = ((G \oplus B)/N)[n] = (G[n] \oplus B[n] + N)/N$. Notice that both $G[n] \oplus B[n] + N$ and $G[n] \oplus 0 + N$ are locally compact since $X[n]$ and $\ker \varphi = (G[n] \oplus 0 + N)/N$ are locally compact ([12, (5.25)]). The group $X[n]/\ker \varphi$ is equal to

$$\frac{(G[n] \oplus B[n] + N)/N}{(G[n] \oplus 0 + N)/N} \cong \frac{G[n] \oplus B[n] + N}{G[n] \oplus 0 + N} = \frac{(0 \oplus B[n]) + (G[n] \oplus 0 + N)}{G[n] \oplus 0 + N}$$

(cf. [12, (5.35)]) and by the second isomorphism theorem in \mathfrak{L} (see [9, Theorem 3.3]), the latter group is isomorphic to

$$\frac{0 \oplus B[n]}{(0 \oplus B[n]) \cap (G[n] \oplus 0 + N)} = \frac{0 \oplus B[n]}{0 \oplus \phi(A[n])} \cong C[n]$$

since $B[n]$ is σ -compact. Thus we have an isomorphism from $X[n]/\ker \varphi$ to $C[n]$ given by $((g, b) + N) + \ker \varphi \mapsto \pi(b)$ ($g \in G[n], b \in B[n]$). Since this map coincides with $\bar{\varphi}$ it follows that $\varphi : X[n] \rightarrow C[n]$ is open. Therefore, αE is $*$ -pure. \square

PROPOSITION 3.4. *Let G be a totally disconnected group in \mathfrak{L} such that $\text{Ext}(\widehat{\mathbb{Q}}, G) = 0$. Then G is a topological torsion group.*

PROOF. By Theorem 2.7, it suffices to show that \widehat{G} is totally disconnected. To prove this, we argue as in the proof of [14, Theorem 2.7 (ii) \Rightarrow (iii)]. First, notice that the quotient $G/B(G)$ is discrete (cf. [12, (9.26)(a)]) and torsion-free, and that $(\mathbb{Q}/\mathbb{Z})^\wedge$ is compact since \mathbb{Q}/\mathbb{Z} is discrete ([12, (23.17)]). The proper exact sequence $0 \rightarrow B(G) \rightarrow G \rightarrow G/B(G) \rightarrow 0$ gives rise to the exact sequence $0 = \text{Ext}(\widehat{\mathbb{Q}}, G) \rightarrow \text{Ext}(\widehat{\mathbb{Q}}, G/B(G)) \rightarrow 0$. But then exactness of the sequence

$$0 = \text{Hom}((\mathbb{Q}/\mathbb{Z})^\wedge, G/B(G)) \rightarrow \text{Ext}(\widehat{\mathbb{Z}}, G/B(G)) \rightarrow \text{Ext}(\widehat{\mathbb{Q}}, G/B(G)) = 0$$

yields $G/B(G) \cong \text{Ext}(\widehat{\mathbb{Z}}, G/B(G)) = 0$ by Proposition 2.3, thus G coincides with $B(G)$. Since \widehat{G}_0 is the annihilator of $B(G)$ in \widehat{G} (cf. [12, (24.17)]), it follows that \widehat{G} is totally disconnected. \square

The following lemma will be needed:

LEMMA 3.5 [14, Lemma 2.6]. *Suppose that G is a group in \mathfrak{L} possessing a compact open subgroup. Then G is densely divisible if and only if G/C is divisible for every compact open subgroup C of G .*

PROPOSITION 3.6. *Suppose that G is a topological torsion group such that $\text{Ext}(X, G) = 0$ for every torsion-free group X in \mathfrak{L} which is either discrete or a topological torsion group. Then G is densely divisible.*

PROOF. Our proof is similar to the second part of the proof of [8, Theorem 7]. Let C be a compact open subgroup of G and set $A = G/C$. Then for any torsion-free group X in \mathfrak{L} which is discrete or a topological torsion group, exactness of the sequence

$$0 = \text{Ext}(X, G) \rightarrow \text{Ext}(X, A) \rightarrow 0$$

yields $\text{Ext}(X, A) = 0$. Recall that a discrete group H is said to be *cotorsion* if $\text{Ext}(J, H) = 0$ for every discrete torsion-free group J (see [6, page 232]). Then the group A is cotorsion. Since A is also torsion, we have $A = B \oplus D$ for some bounded group B and divisible group D (see [6, Corollary 54.4]). A bounded group is a direct sum of cyclic groups ([6, Theorem 17.2]), so if $B \neq 0$, then B contains a direct summand $B' \cong \mathbb{Z}/p^n\mathbb{Z}$ for some prime p and positive integer n . By [13, Example 2.4], there is a non-splitting

proper exact sequence

$$0 \rightarrow B' \rightarrow K \rightarrow L \rightarrow 0$$

in \mathfrak{L} where L is torsion-free and \widehat{L} is a p -group, hence \widehat{L} is a topological torsion group. By Theorem 2.7, L is a topological torsion group and we have $\text{Ext}(L, A) = 0$. But then Theorem 2.4 shows that $\text{Ext}(L, B') = 0$ which is impossible. Therefore $B = 0$ and it follows from Lemma 3.5 that G is densely divisible. \square

Let \mathfrak{C} denote the class of groups X in \mathfrak{L} such that X is connected or X is a torsion-free group which is either discrete or a topological torsion group. Then the groups G in \mathfrak{L} having the property that every $*$ -pure extension of G by a group in \mathfrak{C} splits can be characterized as follows:

THEOREM 3.7. *A group G in \mathfrak{L} satisfies ${}^*\text{Pext}(X, G) = 0$ for all groups X in \mathfrak{C} if and only if $G \cong \mathbb{R}^n \times \mathbb{T}^m$ for some nonnegative integer n and cardinal m .*

PROOF. Sufficiency follows from Theorem 2.1. Conversely, suppose ${}^*\text{Pext}(X, G) = 0$ for all groups X in \mathfrak{C} . By Proposition 3.2, we have $G = G_0 \oplus H$ where $G_0 \cong \mathbb{R}^n \times \mathbb{T}^m$ for some nonnegative integer n and cardinal m . Due to Lemma 3.1(ii), ${}^*\text{Pext}(X, H) = 0$ for all groups X in \mathfrak{C} . Then by Lemma 3.1(i), Proposition 3.4 and Proposition 3.6, H is a densely divisible topological torsion group. By Theorem 2.8, H can be identified with a local direct product of its p -components

$$H_p = \{x \in H : \lim_{n \rightarrow \infty} p^n x = 0\}$$

belonging to different primes p . Assume $H \neq 0$. Then there exists a prime p such that $H_p \neq 0$. Since the projection map $H \rightarrow H_p$ is continuous, H_p is densely divisible, so by Proposition 2.9 it contains a closed subgroup D such that $D \cong F_p$ or $D \cong \mathbb{Z}(p^\infty)$. In either case, D is a divisible σ -compact group in \mathfrak{L} ([12, (10.5)]). For the inclusion map $\alpha : D \rightarrow H$ and a connected group X in \mathfrak{L} , consider the exact sequence

$$0 = \text{Hom}(X, H/D) \rightarrow \text{Ext}(X, D) \xrightarrow{\alpha_*} \text{Ext}(X, H).$$

To show that $\text{Ext}(X, D) = 0$, let $E : 0 \rightarrow D \xrightarrow{\phi} F \rightarrow X \rightarrow 0 \in \text{Ext}(X, D)$. The group $F/\phi(D) \cong X$ is σ -compact since it is connected ([12, (9.14)]) and $\phi(D)$ is σ -compact, hence F is σ -compact ([17, Theorem 6.10(c)]) and it follows that every group $F[n]$ is σ -compact. By Lemma 3.3, $\alpha_*(E) = \alpha E$

is a $*$ -pure extension, so it splits. Since α_* is injective, E splits as well and we obtain $\text{Ext}(X, D) = 0$. But then Theorem 2.6 shows that D is connected, a contradiction. Consequently, $H = 0$ and we have $G \cong \mathbb{R}^n \times \mathbb{T}^m$, as desired. \square

4. Injective and projective properties

A group G in \mathfrak{L} is called *pure injective in \mathfrak{L}* if it has the injective property relative to all pure extensions in \mathfrak{L} , that is, if for every pure proper exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathfrak{L} and every $\alpha \in \text{Hom}(A, G)$ there is a $\beta \in \text{Hom}(B, G)$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\ & & \alpha \downarrow & & \swarrow \beta & & & & \\ & & G & & & & & & \end{array}$$

is commutative. Similarly, we call a group in \mathfrak{L} *$*$ -pure injective in \mathfrak{L}* if it has the injective property relative to all $*$ -pure extensions. Then we have:

THEOREM 4.1. *The following are equivalent for a group G in \mathfrak{L} :*

- (1) G is $*$ -pure injective in \mathfrak{L} ;
- (2) ${}^*\text{Pext}(X, G) = 0$ for all groups X in \mathfrak{L} ;
- (3) $G \cong \mathbb{R}^n \oplus \mathbb{T}^m$ where n is a nonnegative integer and m is a cardinal.

PROOF. Suppose that G is $*$ -pure injective in \mathfrak{L} . Then every $*$ -pure extension $0 \rightarrow G \rightarrow B \rightarrow X \rightarrow 0$ splits because there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & G & \rightarrow & B & \rightarrow & X & \rightarrow & 0 \\ & & \parallel & & \swarrow & & & & \\ & & G & & & & & & \end{array}$$

Consequently, (1) implies (2). By Theorem 3.7, (2) implies (3). The groups of the form $\mathbb{R}^n \times \mathbb{T}^m$ are injective in \mathfrak{L} (Theorem 2.1), hence (3) implies (1). \square

By the theorem above, the $*$ -pure injectives in \mathfrak{L} are exactly the injectives in \mathfrak{L} . As an immediate consequence, we obtain a complete description of the pure injectives in \mathfrak{L} . This extends [14, Theorem 2.7] and shows that the result on discrete and compact injectives in \mathfrak{L} as stated in [7, Proposition 8] is incorrect.

COROLLARY 4.2. *The following are equivalent for a group G in \mathfrak{L} :*

- (1) G is pure injective in \mathfrak{L} ;
- (2) $\text{Pext}(X, G) = 0$ for all groups X in \mathfrak{L} ;
- (3) $G \cong \mathbb{R}^n \oplus \mathbb{T}^m$ where n is a nonnegative integer and m is a cardinal.

Recall that a proper exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathfrak{L} is said to be *topologically pure* if the induced sequence

$$0 \rightarrow \overline{nA} \rightarrow \overline{nB} \rightarrow \overline{nC} \rightarrow 0$$

is proper exact for all positive integers n (see [13]). Pontrjagin duality shows that the sequence E is topologically pure if and only if its dual sequence

$$0 \rightarrow \widehat{C} \rightarrow \widehat{B} \rightarrow \widehat{A} \rightarrow 0$$

is *-pure (see [13, Corollary 2.6]). We call a group G in \mathfrak{L} *topologically pure projective* in \mathfrak{L} if it has the projective property relative to all topologically pure extensions, in other words, if for every topologically pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and every $\gamma \in \text{Hom}(G, C)$ there is a $\delta \in \text{Hom}(G, B)$ such that the diagram

$$\begin{array}{ccccccc}
 & & & & G & & \\
 & & & & \downarrow \gamma & & \\
 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\
 & & & & \delta \nearrow & & \\
 & & & & G & &
 \end{array}$$

is commutative. Then dualization of Theorem 4.1 yields the following result which extends [13, Theorem 4.4(3)]:

COROLLARY 4.3. *The following are equivalent for a group G in \mathfrak{L} :*

- (1) G is topologically pure projective in \mathfrak{L} ;
- (2) every topologically pure sequence $0 \rightarrow A \rightarrow B \rightarrow G \rightarrow 0$ splits;
- (3) $G \cong \mathbb{R}^n \oplus \bigoplus_m \mathbb{Z}$ where n is a nonnegative integer and m is a cardinal.

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