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An A^1 Function that is not in Lip_α for any Positive α

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Abstract

Let A^1 be the Banach algebra of all continuous functions on the torus whose Fourier coefficients are in ℓ^1, and let Lip_α be the Banach algebra of all continuous function f on the torus such that

$$\|f\| = \sup_{x\in\mathbb{T}, h\neq 0} \left|\frac{f(x+h)-f(x)}{h^\alpha}\right| < \infty.$$

We produce an example of an A^1 function that is not in Lip_α for any $0 < \alpha \leq 1$.

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1 Introduction

We define A^1 as the space of all continuous functions on the torus $\mathbb{T}$ with summable Fourier coefficients. This is a well known Banach algebra under pointwise multiplication. Similarily we define A^p to be the space of all continuous functions on $\mathbb{T}$ with p-th power summable Fourier coefficients. Values of p less than one do not yield Banach spaces and because of Plancheral's theorem values of p greater than 2 are all equal to $A^2 = C(\mathbb{T})$ which is a Banach algebra. So we restrict our attention to the range $1 < p < 2$. It has been shown [2] that A^p is not a Banach algebra under pointwise multiplication for any $1 < p < p_0$, for a certain $p_0 \approx 1.18$. One wonders whether or not A^p is an algebra for all p such that $p_0 \leq p < 2$. To answer this question it is natural to compare A^p to other well known algebras of continuous functions on $\mathbb{T}$.

The Lipschitz spaces are a another Banach algebra of continuous functions on the torus. The Lipschitz spaces $\mathrm{Lip}_\alpha, 0 < \alpha \leq 1$ are all continuous functions on the torus with a somewhat weakened version of differentiability depending on the parameter α. More precisely, a continuous function f is in Lip_α if

$$\sup_{x\in\mathbb{T},h\neq 0}\left|\frac{f(x+h)-f(x)}{h^\alpha}\right|<\infty.$$

One may wonder how the Lipschitz algebras are related to the spaces A^p. In one direction there is a very nice inclusion property [1, Theorem 3.2.16], for $1\leq p\leq 2$,

$$f\in \mathrm{Lip}_\alpha,\ \alpha>\frac{2-p}{2p} \text{ implies } f\in A^p.$$

However, there is no hope for any kind of reverse inclusion due to the following theorem.

2 Main Theorem

Theorem 2.1. *There exists a function in A^1 that is not in Lip_α for any $0<\alpha\leq 1$.*

Proof. Let

$$f(x)=\sum_{n=1}^{\infty} n^{-(1+\delta)}e^{in^\epsilon}e^{inx}$$

where $0<\delta,\epsilon<1$. The function f is a continuous function on the torus with summable Fourier coefficients, and furthermore $f\notin \mathrm{Lip}_\alpha$ for all $\alpha>\frac{1}{2}\epsilon+\delta$, see [3, Theorem 5.2, page 200]. For all $m>1$, define

$$g_m(x)=\sum_{n=1}^{\infty} n^{-(1+\frac{1}{2m})}e^{in^{1/m}}e^{inx}.$$

So we see $g_m\in A^1$ for all m, and $g_m\notin \mathrm{Lip}_\alpha$ for all $\alpha>\frac{1}{m}$. We use these functions g_m to produce $g=\sum_{m=2}^{\infty}\lambda_m g_m$ such that $g\in A^1$ but $g\notin \mathrm{Lip}_\alpha$ for all $\alpha>0$. To do this, we produce recursively a sequence $\{(\lambda_m,\mu_m,x_m,h_m)\}_{m=2}^{\infty}$ to satisfy

1. $0<\lambda_m\leq m^{-2}\|g_m\|_{A^1}^{-1}$
2. $\lambda_m\mu_m-\sum_{k<m}\lambda_k\|g_k\|_{\mathrm{Lip}_{\frac{1}{m-1}}}>m+1$
3. $0<|h_m|<1$
4. $|g_m(x_m+h_m)-g_m(x_m)|/|h_m|^{1/(m-1)}>\mu_m$
5. $\lambda_m\cdot 2\|g_m\|_\infty/|h_k|^{1/(k-1)}<2^{-m}$ for $2\leq k<m$.

Choose λ_2 to satisfy (1) and μ_2 so that $\lambda_2\mu_2 > 3$, thus satisfying (2). Then x_2 and h_2 to satisfy (3) and (4). Now, if $(\lambda_m, \mu_m, x_m, h_m)$ have been chosen for $2 \le m < N$, for some $N \in \mathbb{Z}^+$, choose λ_N to satisfy (1) and (5), then μ_N to satisfy (2), and finally x_N and h_N to satisfy (3) and (4).

By (1) the series $\sum_{m=2}^{\infty} \lambda_m g_m$ converges in A^1 to a function $g \in A^1$. Suppose $0 < \alpha < 1$ is given. Take an integer $m > 1 + \frac{1}{\alpha}$, so $\alpha > \frac{1}{m-1}$. Then

$$\begin{aligned}
\|g\|_{\mathrm{Lip}_\alpha} &\ge \frac{|g(x_m + h_m) - g(x_m)|}{|h_m|^\alpha} \\
&\ge \frac{|g(x_m + h_m) - g(x_m)|}{|h_m|^{\frac{1}{m-1}}} \\
&\ge \lambda_m \frac{|g_m(x_m + h_m) - g_m(x_m)|}{|h_m|^{\frac{1}{m-1}}} - \sum_{k<m} \lambda_k \frac{|g_k(x_m + h_m) - g_k(x_m)|}{|h_m|^{\frac{1}{m-1}}} \\
&\quad - \sum_{k>m} \lambda_k \frac{|g_k(x_m + h_m) - g_k(x_m)|}{|h_m|^{\frac{1}{m-1}}} \\
&\ge \lambda_m \mu_m - \sum_{k<m} \lambda_k \|g_k\|_{\mathrm{Lip}_{\frac{1}{m-1}}} - \sum_{k>m} \lambda_k \frac{2\|g_k\|_\infty}{|h_m|^{\frac{1}{m-1}}} \\
&> m + 1 - \sum_{k>m} 2^{-k} \\
&> m.
\end{aligned}$$

Since $\|g\|_{\mathrm{Lip}_\alpha} > m$ for all $m > 1 + \frac{1}{\alpha}$, $g \notin \mathrm{Lip}_\alpha$. Thus we have a continuous function with a summable sequence of Fourier coefficients that is not in any Lipschitz class.

□

If one thinks of the Lipschitz algebras as being slightly strengthened versions of continuous functions, then A^1 is a space of continuous functions where the notion of continuity cannot be strengthened for at least one member. In fact, one has the following theorem.

Theorem 2.2. *The set of all functions, S, in A^1 that are not in* Lip_α *for all positive α is dense in A^1.*

Proof. Let $\epsilon > 0$, and let f be an A^1 function that is not in S. Hence there exists an $\alpha > 0$ such that $f \in \mathrm{Lip}_\alpha$. Let $h = f + \frac{\epsilon}{M} g$, where g is the function from the previous theorem and M is a number such that $\|g\|_{A^1} < M$. The function h is in S, furthermore

$$\|f - h\|_{A^1} < \epsilon.$$

Therefore S is dense in A^1. □

References

[1] Loukas Grafakos, *Classical and Modern Fourier Analysis,* Pearson Prentice-Hall, New Jersey, 2004.

[2] Ryan Mullen, *Examples of Banach Spaces that are not Banach Algebras,* Function spaces (Edwardsville, IL, 2006), Contemp. Math., vol. 435, Amer. Math. Soc., Providence, RI, 2007, pp. 335-342.

[3] A. Zygmund, *Trigonometric series. Vol. I, II,* Cambridge University Press, Cambridge, 1977.

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