8-30-1996

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ELSEVIER Theoretical Computer Science 163 (1996) 177–192

Theoretical Computer Science

Frequency computation and bounded queries

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Received February 1995; revised July 1995
Communicated by R.V. Book

Abstract

There have been several papers over the last ten years that consider the *number of queries* needed to compute a function as a measure of its complexity. The following function has been studied extensively in that light: $\mathrm{F}_a^A(x_1,\ldots,x_a) = A(x_1)\cdots A(x_a)$. We are interested in the complexity (in terms of the number of queries) of *approximating* F_a^A. Let $b \leqslant a$ and let f be any function such that $\mathrm{F}_a^A(x_1,\ldots,x_a)$ and $f(x_1,\ldots,x_a)$ agree on at least b bits. For a general set A we have matching upper and lower bounds on f that depend on coding theory. These are applied to get exact bounds for the case where A is semirecursive, A is superterse, and (assuming $\mathrm{P} \neq \mathrm{NP}$) $A = \mathrm{SAT}$. We obtain exact bounds when A is the halting problem using different methods.

1. Introduction

The complexity of a function can be measured by the number of queries (to some oracle) needed to compute it. This notion has been studied in both a recursion-theoretic framework (see, for example, [5, 11, 17]) and a complexity-theoretic framework (see, for example, [2, 12, 16]). We give several examples.

1. Let f be the function that, given a graph on n vertices, outputs the number of colors needed to color it. Krentel [16] showed that this function can be computed with $\mathrm{O}(\log n)$ queries to SAT in polynomial time but cannot be computed with substantially fewer queries to any oracle in polynomial time (unless $\mathrm{P} = \mathrm{NP}$).

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* Corresponding author. Email: gasarch@cs.umd.edu. Supported in part by NSF grants CCR-8803641 and CCR-9020079.

[1] Supported in part by National Science Foundation grant CCR-8958528

[2] Supported in part by NSF

0304-3975/96/$15.00
SSDI 0304-3975(95)00149-2

2. Let A be a nonrecursive set and $a \in \mathcal{N}$. Let $\#_a^A$ be the function that, given $(x_1,\ldots,x_a)$, returns $|A \cap \{x_1,\ldots,x_a\}|$ (the number of elements that are in A). It is known that there are sets A,X such that $\#_a^A$ can be computed with $\lceil\log(a+1)\rceil - 1$ queries to X. Kummer [17] showed that this is optimal, i.e., if $\#_a^A$ can be computed with $\lceil\log(a+1)\rceil$ queries to some X then A is recursive.

The following functions have been studied extensively in this light.

Definition 1.1. Let $a \in \mathcal{N}$ and $A \subseteq \mathcal{N}$. The function $\mathrm{F}_a^A : \mathcal{N}^a \to \{0,1\}^a$ is defined as

$$\mathrm{F}_a^A(x_1,\ldots,x_a) = A(x_1)\cdots A(x_a).$$

The function $\#_a^A$ is defined as

$$\#_a^A(x_1,\ldots,x_a) = |A \cap \{x_1,\ldots,x_a\}|.$$

The function F_a^A is interesting because it has a certain intuitive appeal and most lower bounds have reduced to lower bounds for F_a^A. We investigate the complexity of computing an approximation to F_a^A. To do this we define a class of functions $\mathit{freq}_{b,a}^A$ such that every element of $\mathit{freq}_{b,a}^A$ approximates F_a^A.

Notation. If σ,τ are strings of the same length then $\sigma =^c \tau$ means that σ and τ differ in at most c places.

Definition 1.2. Let $a,b \in \mathcal{N}$ be such that $1 \leqslant b \leqslant a$, and let $A \subseteq \mathcal{N}$. $\mathit{freq}_{b,a}^A$ is the set of all functions f that map $\mathcal{N}^a$ to $\{0,1\}^a$ such that, for all $x_1,\ldots,x_a$, $f(x_1,\ldots,x_a)$ and $\mathrm{F}_a^A(x_1,\ldots,x_a)$ agree in at least b places (i.e., $f(x_1,\ldots,x_a) =^{a-b} \mathrm{F}_a^A(x_1,\ldots,x_a)$). We occasionally treat $\mathit{freq}_{b,a}^A$ as just one function: an upper bound on the complexity of $\mathit{freq}_{b,a}^A$ means *at least one* function in $\mathit{freq}_{b,a}^A$ has that complexity (or less), and a lower bound on the complexity of $\mathit{freq}_{b,a}^A$ means that *every* functions in $\mathit{freq}_{b,a}^A$ has that complexity (or greater).

Note. The set $\mathit{freq}_{b,a}^A$ was first defined by Rose [22] and has a long history. For more information see [13].

We investigate the complexity of $\mathit{freq}_{b,a}^A$ for several sets (or types of sets) A and parameters a,b. Our measure of complexity of a function is the number of queries needed to compute it. Most of our results are recursion-theoretic; however, some of our techniques also apply in a polynomial framework.

Information about the complexity of F_a^A will help in our study. However, the complexity of $\mathit{freq}_{b,a}^A$ is a harder question. We describe the difference. Assume that, given $(x_1,\ldots,x_a)$, one could produce (the index for) an r.e. set $W \subseteq \{0,1\}^a$ such that $\mathrm{F}_a^A(x_1,\ldots,x_a) \in W$. It has been shown (Lemma 2.4) that the size of W *completely* determines the complexity of F_a^A. Does knowing W help us to compute $\mathit{freq}_{b,a}^A(x_1,\ldots,x_a)$? From W we can obtain W', the set of vectors that differ from elements of W by at most $a-b$ bits. Formally,

$$W' = \{\vec{v} : (\exists \vec{c} \in W)[\vec{v} =^{a-b} \vec{c}]\}.$$

It is easy to see that $freq^A_{b,a}(x_1,\dots,x_a) \in W'$. The complexity of $freq^A_{b,a}$ is completely determined by $|W'|$. Unfortunately, it is impossible to determine $|W'|$ from $|W|$. To determine $|W'|$ we need to know the very *structure* of W. This is the key reason that F^A_a is better understood than $freq^A_{b,a}$: the complexity of F^A_a is related to *the cardinality of* W, while the complexity of $freq^A_{b,a}$ is related to *the structure of* W. One theme of this paper will be that the more we know about W the better we understand the complexity of $freq^A_{b,a}$.

In Section 3 we prove a general lower bound on the complexity of $freq^A_{b,a}$ (for nonrecursive A). It is based on a general lower bound for $\#^A_a$. In Section 4 we obtain exact bounds for the complexity of $freq^K_{b,a}$. In Section 5 we link the complexity of $freq^A_{b,a}$ to the structure of the set W mentioned above. This will allow us to establish the exact complexity of $freq^A_{b,a}$ for certain sets A. These exact complexities depend on functions from coding theory. In Section 6 we use our proof techniques to obtain results in complexity theory. Assuming $\mathrm{P} \neq \mathrm{NP}$ we determine the exact query complexity of $freq^{\mathrm{SAT}}_{b,a}$.

2. Definitions, conventions and useful lemmas

Notation. We use the following notation throughout this paper.

1. $M_0, M_1, \dots$ is a standard effective list of Turing machines.
2. $M_0^{()}, M_1^{()}, \dots$ is a standard effective list of oracle Turing machines.
3. W_e is the domain of M_e. Hence, $W_0, W_1, \dots$ is an effective list of all r.e. sets.
4. $K = \{e : M_e(e)\downarrow\}$.
5. If $A \subseteq \mathcal{N}$ then $A' = \{e : M_e^A(e)\downarrow\}$.
6. $D_e = \{i :$ the ith bit of e is $1\}$. Hence $D_0, D_1, \dots$ is a list of all finite sets.

Convention. Technically, M_e takes elements of $\mathcal{N}$ as input and returns elements of $\mathcal{N}$ as output; and $W_e, D_e \subseteq \mathcal{N}$. We will sometimes need to use $\mathcal{N}^a$ (or $\{0,1\}^*$) instead of $\mathcal{N}$. In these cases we implicitly assume that there is a fixed recursive bijection between $\mathcal{N}$ and $\mathcal{N}^a$ ($\{0,1\}^*$) and code elements of $\mathcal{N}^a$ ($\{0,1\}^*$) into $\mathcal{N}$ accordingly.

Definition 2.1. Let $a \in \mathcal{N}$ and let $X \subseteq \mathcal{N}$. $\mathrm{FQ}(a,X)$ is the collection of all total functions g such that g is recursive in X via an algorithm that makes at most a sequential queries to X. $\mathrm{FQC}(a,X)$ is the collection of all functions g such that g is recursive in X via an algorithm $M^{()}$ such that (1) for all x, $M^X(x)$ makes at most a sequential queries to X, and (2) for all x, Y the computation $M^Y(x)$ converges.

The concept of bounded queries is tied to enumerability. Every possible sequence of query answers leads to a possible answer. Hence, less answers are possible with fewer queries.

Definition 2.2. Let $a \in \mathcal{N}$ and f be any total function. The function f is *a-enumerable*, and we write $f \in \mathrm{EN}(a)$, if there exists a recursive function g such that, for all x, $|W_{g(x)}| \leqslant a$ and $f(x) \in W_{g(x)}$. (This concept first appeared in a recursion-theoretic framework in [3]. The name "enumerable" is from [7] where it was defined in a polynomial bounded framework.)

If f is a-enumerable then, given x, we can find $g(x)$ and try to enumerate $W_{g(x)}$ looking for possibilities for $f(x)$. While doing this we do not know when $W_{g(x)}$ will have stopped generating possibilities. The next definition imposes a stronger condition of enumeration. In this scenario we are given an index of a set of possibilities as an index of a finite set. Hence, we can obtain all the possibilities and know we have them all.

Definition 2.3. Let $a \in \mathcal{N}$ and f be any total function. The function f is *strongly a-enumerable*, and we write $f \in \mathrm{SEN}(a)$, if there exists a recursive function g such that, for all x, $|D_{g(x)}| \leqslant a$ and $f(x) \in D_{g(x)}$. We denote this by $f \in \mathrm{SEN}(a)$.

Lemma 2.4 (Beigel [3, 5]). *Let $a \in \mathcal{N}$ and let f be any function.*

1. $(\exists X)[f \in \mathrm{FQ}(a,X)]$ *iff* $f \in \mathrm{EN}(2^a)$.
2. $(\exists X)[f \in \mathrm{FQC}(a,X)]$ *iff* $f \in \mathrm{SEN}(2^a)$.

In this paper we will prove upper and lower bounds in terms of enumerability (or strong enumerability). Using Lemma 2.4 the reader can obtain corollaries about upper and lower bounds in terms of number of queries.

The following lemma provides a lower bound on the enumerability of $\#_a^A$. We will use it in Theorem 3.1 to obtain a lower bound on $freq_{b,a}^A$.

Lemma 2.5 (Kummer [17]). *Let $a \in \mathcal{N}$, and let $A \subseteq \mathcal{N}$. If $\#_a^A \in \mathrm{EN}(a)$, then A is recursive.*

We now exhibit nonrecursive sets A (namely the semirecursive sets) such that if $b/a \leqslant \frac{1}{2}$ then $freq_{b,a}^A$ is recursive. Since we are interested in how many queries are required to compute $freq_{b,a}^A$, the case where $freq_{b,a}^A$ is recursive is not of interest. Hence, most of our theorems will assume $b/a > \frac{1}{2}$.

Definition 2.6 (Jockusch [15]). A set A is *semirecursive* if there exists a recursive linear ordering $\sqsubset$ on $\mathcal{N}$ such that A is closed downward under $\sqsubset$. (This definition is equivalent to the following: A is semirecursive if there exists a total recursive function f such that $A \cap \{x,y\} \neq \emptyset \Rightarrow f(x,y) \in A \cap \{x,y\}$. The proof of the equivalence is in [15] and credited to Appel and McLaughin.)

The following is a folk theorem. It will also be a consequence of Theorem 5.10.

Proposition 2.7. *Assume $b/a \leqslant \frac{1}{2}$. If A is semirecursive then $freq_{b,a}^A$ is recursive. Hence, every tt-degree contains a set A such that $freq_{b,a}^A$ is recursive.*

Proof. Let A be semirecursive via $\sqsubset$. Given $(x_1,\ldots,x_a)$ we may assume $x_1 \sqsubset \cdots \sqsubset x_a$. Since $\mathrm{F}^A_a(x_1,\ldots,x_a) \in \{1^i0^{a-i} : 0 \leqslant i \leqslant a\}$ we have $1^{\lceil a/2 \rceil}0^{\lfloor a/2 \rfloor} =^{a-b} \mathrm{F}^A_a(x_1,\ldots,x_a)$. Output $1^{\lceil a/2 \rceil}0^{\lfloor a/2 \rfloor}$.

Part 2 follows from part 1 since Jockusch [15] showed that every tt-degree contains a semirecursive set. □

It is known that Proposition 2.7 is optimal; if $b/a > \frac{1}{2}$ and $freq^A_{b,a}$ is recursive then A is recursive. This was proven by Trakhtenbrot [25]. We will give an alternative proof (Corollary 3.2).

3. A general lower bound for $freq^A_{b,a}$

We prove a general lower bound on the enumerability of $freq^A_{b,a}$ for any non-recursive A.

Theorem 3.1. *Assume* $1 \leqslant b \leqslant a$, $b/a > \frac{1}{2}$, *and* $A \subseteq \mathcal{N}$. *If* $freq^A_{b,a} \cap \mathrm{EN}(\lceil (a+1)/(2(a-b)+1) \rceil - 1) \neq \emptyset$, *then A is recursive.*

Proof. Assume that $f \in freq^A_{b,a} \cap \mathrm{EN}(\lceil (a+1)/(2(a-b)+1) \rceil - 1)$. Let $(x_1,\ldots,x_a) \in \mathcal{N}^a$. Every time a possibility for $f(x_1,\ldots,x_a)$ is generated it yields at most $2(a-b)+1$ possibilities for $\#^A_a(x_1,\ldots,x_a)$. Hence,

$$\#^A_a \in \mathrm{EN}\left(\left(\left\lceil \frac{a+1}{2(a-b)+1} \right\rceil - 1\right)(2(a-b)+1)\right) \subseteq \mathrm{EN}(a).$$

By Lemma 2.5 A is recursive. □

Corollary 3.2 (Trakhtenbrot [25]). *If $b/a > \frac{1}{2}$ and $freq^A_{b,a}$ is recursive, then A is recursive.*

Note. Theorem 3.1 has been obtained independently by Kummer and Stephan [19] using different methods.

The next theorem shows that Theorem 3.1 cannot be improved, and also extends Proposition 2.7.

Theorem 3.3. *Assume* $1 \leqslant b \leqslant a$, $b/a > \frac{1}{2}$. *If A is semirecursive then*

$$freq^A_{b,a} \cap \mathrm{SEN}(\lceil (a+1)/(2(a-b)+1) \rceil) \neq \emptyset.$$

Proof. Let $k = \lceil (a+1)/(2(a-b)+1) \rceil$. We present an algorithm for a function $f \in freq^A_{b,a} \cap \mathrm{SEN}(k)$.

Assume the input is $x_1,\ldots,x_a$. We can assume that $x_1 \sqsubset \cdots \sqsubset x_a$. Hence $\mathrm{F}^A_a(x_1,\ldots,x_a) \in S = \{1^c0^{a-c} : 0 \leqslant c \leqslant a\}$. For $1 \leqslant i \leqslant k-1$ let $v_i = 1^{(2i-1)(a-b)+i-1}0^{a-(2i-1)(a-b)-i+1}$, and let $v_k = 1^b0^r$. Let $f(x_1,\ldots,x_a)$ be an index for the finite set $D = \{v_1,\ldots,v_k\}$. It is easy to check that for every $w \in S$ there exists $v \in D$ such that $w =^{a-b} v$. □

4. Exact bounds for $freq^K_{b,a}$

In this section we determine the *exact* complexity of $freq^K_{b,a}$ in terms of enumerability. In Corollary 5.19 we will determine the *exact* complexity of $freq^K_{b,a}$ in terms of strong enumerability. It is known that $\#^K_a(x_1,\dots,x_a)$ completely determines F^K_a. Hence, the structure of the set of possibilities for F^K_a is well understood. This is why we are able to obtain exact bounds.

Theorem 4.1. *If* $1 \leqslant b \leqslant a$ *then* $freq^K_{b,a} \cap \mathrm{EN}(\lceil (a+1)/((a-b)+1)\rceil) \neq \emptyset$.

Proof. Given $(x_1,\dots,x_a)$ we show how to enumerate $\leqslant \lceil (a+1)/((a-b)+1)\rceil$ possibilities such that one of them agrees with $\mathrm{F}^K_a(x_1,\dots,x_a)$ on at least b positions.

Let $k = \lceil (a+1)/((a-b)+1)\rceil$, and let $I_1,\dots,I_k$ be intervals of length at most $a-b+1$ that partition $\{0,\dots,a\}$. (Notice that $k > 1$ because $b \geqslant 1$.) For each interval $I = [c,d]$ we enumerate a possibility that is based on the belief that $\#^K_a(x_1,\dots,x_a) \in [c,d]$. By dovetailing these computations we enumerate at most k possibilities.

For interval $I = [c,d]$ we do the following. If $c = 0$ then output $(0,\dots,0)$. If $c > 0$ then simultaneously run all of $M_{x_1}(x_1),\dots,M_{x_a}(x_a)$ until exactly c of them halt (this need not happen). Output a string that indicates that these c programs are in K and no other programs are in K.

We show that if $\#^K_a(x_1,\dots,x_a) \in I = [c,d]$ then the possibility associated with I is correct. Clearly, the c 1's are correct. Since there are at most d programs in K, at least $a-d$ of the 0's are correct. Hence, at least $c+a-d = a+(c-d) = a+1-|I| \geqslant a+1-(a-b+1) = b$ bits are correct. □

Note. By Lemma 2.4, $(\exists X)[freq^K_{b,a} \cap \mathrm{FQ}(\lceil \log(a+1)/((a-b)+1)\rceil, X) \neq \emptyset]$. The oracle is unspecified. In this case we can do just as well with oracle K: by a truncated binary search, $freq^K_{b,a} \cap \mathrm{FQ}(\lceil \log(a+1)/((a-b)+1)\rceil, K) \neq \emptyset$.

The enumeration procedure used in Theorem 4.1 is not a strong enumeration. In Section 5 we show that a strong enumeration for $freq^K_{b,a}$ requires many more possibilities than an enumeration.

We show that the above bound is tight. For this we need the a-ary recursion theorem which we state carefully. Smullyan ([23], see also [21, p. 190]) proved this for $a = 2$ but the general case is an easy extension.

Proposition 4.2. *Let* $a \geqslant 1$. *For any finite sequence* $g_1,\dots,g_a$ *of total recursive functions there exists* $x_1,\dots,x_a$ *such that*

$$\varphi_{x_i} = \varphi_{g_i(\langle x_1,\dots,x_a\rangle)}$$

for every $1 \leqslant i \leqslant a$.

Note 4.3. Note that program x_i can use the numbers $x_1,\dots,x_a$. In this sense we think of φ_{x_i} as "knowing" $x_1,\dots,x_a$.

Theorem 4.4. *If* $1 \leqslant b \leqslant a$ *then* $\mathit{freq}^K_{b,a} \cap \mathrm{EN}(\lceil (a+1)/((a-b)+1) \rceil - 1) = \emptyset$.

Proof. Assume, by way of contradiction, that there exists

$$f \in \mathit{freq}^K_{b,a} \cap \mathrm{EN}(\lceil (a+1)/((a-b)+1) \rceil - 1).$$

Assume that $f \in \mathrm{EN}(\lceil (a+1)/((a-b)+1) \rceil - 1)$ via g. We create programs $x_1, \ldots, x_a$ that conspire to cause

$$(\forall \vec{v} \in W_{g(x_1,\ldots,x_a)})[\neg(\vec{v} =^{a-b} \mathrm{F}^K_a(x_1, \ldots, x_a))].$$

We plan to have different blocks of programs invalidate different elements of $W_{g(x_1,\ldots,x_a)}$. Let $k = \lceil (a+1)/((a-b)+1) \rceil - 1$. Since $b \geqslant 1$ we have $k \geqslant 1$. Let $J_1, \ldots, J_k$ be intervals of length $\geqslant a - b + 1$ that partition $\{0, \ldots, a\}$.

By the a-ary recursion theorem we can assume that x_i has access to the numbers $\{x_1, \ldots, x_a\}$.

ALGORITHM FOR x_i

1. Let j be such that $i \in J_j$ (if no such j exists then diverge).
2. Enumerate $W_{g(x_1,\ldots,x_a)}$ until j elements appear (this step might not terminate). Let that jth element be $\vec{v} = b_1 \cdots b_a$.
3. If $b_i = 0$ then converge. If $b_i = 1$ then diverge.

END OF ALGORITHM

For all j, $1 \leqslant j \leqslant k$, if $W_{g(x_1,\ldots,x_a)}$ has the jth element $\vec{v}$, then $\vec{v}$ and $\mathrm{F}^K_a(x_1, \ldots, x_a)$ differ on the bits specified by J_j. Hence, they differ on at least $a - b + 1$ places, so $(\forall \vec{v} \in W_{g(x_1,\ldots,x_a)})[\neg(\vec{v} =^{a-b} \mathrm{F}^K_a(x_1, \ldots, x_a))]$. □

5. Exact bounds for $\mathit{freq}^A_{b,a}$

In this section we prove a general theorem relating the complexity of $\mathit{freq}^A_{b,a}$ to the structure of the set of possible values for F^A_a. We apply this theorem to semirecursive sets, joins of semirecursive sets, and superterse sets.

The following definitions from coding theory are used extensively in this section.

Definition 5.1. Let $a, r \in \mathcal{N}$. Let $z \in \{0,1\}^a$. The *closed ball of radius r centered at* z is the set $B(z,r) = \{y \in \{0,1\}^a : y =^r z\}$. If $D \subseteq \{0,1\}^a$ then D *is covered by k balls of radius r* means that there exist $z_1, \ldots, z_k$ such that $D \subseteq \bigcup_{i=1}^k B(z_i, r)$.

Definition 5.2. Let $a, r \in \mathcal{N}$ and $D \subseteq \{0,1\}^a$. Define $k(D,r)$ to be the minimal number j such that D can be covered by j balls of radius r. The quantity $k(\{0,1\}^a, r)$ is denoted by $k(a,r)$.

The quantity $k(a,r)$ is known as *the covering number*. It has been studied extensively (see [8–10, 14, 26]). No exact formula is known for it, however we present some known estimates.

Fact 5.3. *Let* $S_{a,r} = \sum_{i=0}^{r} \binom{a}{i}$.

1. $2^a/S_{a,r} \leqslant k(a,r) \leqslant (2^a/S_{a,r})(1+\log S_{a,r})$ [8, Theorem 3]. (*Better lower bounds are known* [26, Theorem 10].)
2. $k(r+1,r) = k(r+2,r) = \cdots = k(2r+2,r) = 2$ [10, Theorem 14].
3. $k(2r+3,r) = 3$, *and* $7 \leqslant k(2r+4,r) \leqslant 12$ [10, Theorem 14].

Definition 5.4. Let $a, r \in \mathcal{N}$ and $\mathscr{D} \subseteq 2^{\{0,1\}^a}$. We define $k(\mathscr{D},r)$ to be $\max\{k(D,r) : D \in \mathscr{D}\}$.

We now define the notions of $\mathscr{D}$-verbose and strongly $\mathscr{D}$-verbose in order to state a very general result. Note that every set is strongly $2^{\{0,1\}^a}$-verbose.

Definition 5.5. Let $a \in \mathcal{N}$. Let $\mathscr{D} \subseteq 2^{\{0,1\}^a}$. A set A is *$\mathscr{D}$-verbose* if there is a recursive function g such that, for all $x_1,\ldots,x_a$, $W_{g(x_1,\ldots,x_a)} \in \mathscr{D}$ and $\mathrm{F}_a^A(x_1,\ldots,x_a) \in W_{g(x_1,\ldots,x_a)}$. A set A is *strongly $\mathscr{D}$-verbose* if there is a recursive function g such that, for all $x_1,\ldots,x_a$, $D_{g(x_1,\ldots,x_a)} \in \mathscr{D}$ and $\mathrm{F}_a^A(x_1,\ldots,x_a) \in D_{g(x_1,\ldots,x_a)}$.

The following theorem provides for any $A \subseteq \mathcal{N}$ (1) matching upper and lower bounds for the strong enumerability of $freq_{b,a}^A$, and (2) lower bounds for the enumerability of $freq_{b,a}^A$. All results in this paper, except those involving $freq_{b,a}^K$, follow from it.

Theorem 5.6. *Assume* $1 \leqslant b \leqslant a$ *and* $A \subseteq \mathcal{N}$. *For all* k *the following hold.*

(1) *The following are equivalent.*
  (a) *There exists* $\mathscr{D} \subseteq 2^{\{0,1\}^a}$ *such that* A *is strongly* $\mathscr{D}$*-verbose and* $k(\mathscr{D},a-b) \leqslant k$.
  (b) $freq_{b,a}^A \cap \mathrm{SEN}(k) \neq \emptyset$.

(2) *If* $freq_{b,a}^A \cap \mathrm{EN}(k) \neq \emptyset$ *then there exists* $\mathscr{D} \subseteq 2^{\{0,1\}^a}$ *such that* A *is* $\mathscr{D}$*-verbose and* $k \geqslant k(\mathscr{D},a-b)$.

Proof. (1)(a) $\Rightarrow$ (b): Assume A is strongly $\mathscr{D}$-verbose via g. Given $(x_1,\ldots,x_a)$ we strongly enumerate $\leqslant k$ possibilities one of which must agree with $\mathrm{F}_a^A(x_1,\ldots,x_a)$ on at least b positions. Find $D = D_{g(x_1,\ldots,x_a)}$. Find a set of vectors $\{\vec{v}_1,\ldots,\vec{v}_k\}$ such that $D \subseteq \bigcup_{i=1}^k B(\vec{v}_i, a-b)$. (Such vectors exist since $k(\mathscr{D},a-b) \leqslant k$.) Enumerate $\vec{v}_1,\ldots,\vec{v}_k$ as possibilities. Since $\mathrm{F}_a^A(x_1,\ldots,x_a) \in D$

$$(\exists i)[\mathrm{F}_a^A(x_1,\ldots,x_a) \in B(\vec{v}_i, a-b)]$$

so

$$(\exists i)[\mathrm{F}_a^A(x_1,\ldots,x_a) =^{a-b} \vec{v}_i].$$

(1)(b) $\Rightarrow$ (a): Assume $freq_{b,a}^A \cap \mathrm{SEN}(k) \neq \emptyset$. Then there exist k total recursive functions $p_1,\ldots,p_k$ such that $(\forall x_1,\ldots,x_a)(\exists i)[p_i(x_1,\ldots,x_a) =^{a-b} \mathrm{F}_a^A(x_1,\ldots,x_a)]$.

Let

$$D_{g(x_1,\dots,x_a)} = \bigcup_{i=1}^{k} B(p_i(x_1,\dots,x_a), a-b),$$

$$\mathscr{D} = \{D_{g(x_1,\dots,x_a)} : x_1,\dots,x_a \in \mathscr{N}\}.$$

Clearly, A is strongly $\mathscr{D}$-verbose. Since every element of $\mathscr{D}$ is a union of k balls of radius $a-b$, $k \geqslant \max\{k(D,a-b) : D \in \mathscr{D}\}$.

(2) Similar to the proof of part (1)(b) $\Rightarrow$ (a). □

Note 5.7. The converse of Theorem 5.6.2 is not known to be true. The proof of part (1)(a) $\Rightarrow$ (b), cannot be used. In that proof, since A is strongly $\mathscr{D}$-verbose, we are able to find $D \in \mathscr{D}$ and then find its covering set. If A was merely $\mathscr{D}$-verbose then we need not ever really have D, only a subset of D. From this subset it may be impossible to deduce what D really is.

Theorem 5.6 yields matching upper and lower bounds; however, they are not readily computable. The following lemma will be helpful in computing them.

Lemma 5.8. *Let $a, r \in \mathscr{N}$ and $A \subseteq \mathscr{N}$.*

1. *If there exists $\mathscr{D}$ such that A is strongly $\mathscr{D}$-verbose and $k = k(\mathscr{D}, r)$ then $\#_a^A \in$ SEN$(k \cdot (2r+1))$.*
2. *If there exists $\mathscr{D}$ such that A is (strongly) $\mathscr{D}$-verbose then F_a^A is (strongly) $\max\{|D| : D \in \mathscr{D}\}$-enumerable.*

Proof. (1) Assume A is strongly $\mathscr{D}$-verbose via g. We show how to $k(2r+1)$-enumerate $\#_a^A$. On input $(x_1,\dots,x_a)$ find $D = D_{g(x_1,\dots,x_a)}$. We know D can be covered by k balls of radius r. Let $\vec{v}_1,\dots,\vec{v}_k$ be the centers of those balls. Let a_i be the number of 1's in $\vec{v}_i$. Enumerate

$$\{a_i + a : 1 \leqslant i \leqslant k \text{ and } -r \leqslant a \leqslant r\}.$$

These are the $k(2r+1)$ numbers one of which must be $\#_a^A(x_1,\dots,x_a)$.

(2) This follows from the definition of (strongly) $\mathscr{D}$-verbose. □

Note. Kummer and Stephan [18, Corollary 4.3,4.4] have found a different connection between covering numbers and $freq_{b,a}^A$. Let $\Omega(b,a) = \{A : freq_{b,a}^A$ is recursive$\}$. They have shown the following.

1. $(\forall a \geqslant 2)(\exists A, A$ 2-r.e.$)[A \in \Omega(1, \lceil \log(k(a,1)+1) \rceil) - \Omega(2,a)]$.
2. $(\forall b \geqslant 2)(\exists A, A$ r.e.$)[A \in \Omega(1, 2^b - b) - \Omega(2, 2^b - 1)]$.

5.1. Semirecursive sets

We established matching upper and lower bounds for $freq_{b,a}^A$ when A is semirecursive using Proposition 2.7 and Theorems 3.1 and 3.3. Here we give an alternative proof using our general theorem.

Lemma 5.9. *Let* $D = \{1^i0^{a-i} : 0 \leqslant i \leqslant a\}$, *and let* $0 \leqslant r \leqslant a$. *Then* $k(D,r) = \lceil (a+1)/(2r+1) \rceil$.

Proof. Let $k = \lceil (a+1)/(2r+1) \rceil$. For $1 \leqslant i \leqslant k-1$, let $z_i = 1^{(2i-1)r+i-1}0^{a-(2i-1)r-i+1}$, and let $z_k = 1^{a-r}0^r$. It is easy to check that $D \subseteq \bigcup_{i=1}^{k} B(z_i, r)$. Hence $k(D,r) \leqslant k$.

If $\leqslant k-1$ balls of radius r are used then $\leqslant (k-1)(2r+1) \leqslant a$ elements are covered. Hence $k(D,r) \geqslant k$.

Combining the inequalities we obtain $k(D,r) = k$. □

Theorem 5.10. *Assume* $1 \leqslant b \leqslant a$, *A is a semirecursive set that is not recursive, and* $k = \lceil (a+1)/(2(a-b)+1) \rceil$. *Then* $freq^A_{b,a} \cap \mathrm{SEN}(k) \neq \emptyset$ *but* $freq^A_{b,a} \cap \mathrm{SEN}(k-1) = \emptyset$. *Note that if* $b/a \leqslant \frac{1}{2}$ *then* $k = 1$ *so* $freq^A_{b,a} \cap \mathrm{EN}(1) \neq \emptyset$, *hence some function in* $freq^A_{b,a}$ *is recursive.*

Proof. Let A be a semirecursive set with ordering $\sqsubset$. Let $D = \{1^i0^{a-i} : 0 \leqslant i \leqslant a\}$. Let $\mathscr{D}$ be the singleton set $\{D\}$. Semirecursive sets are strongly $\mathscr{D}$-verbose: on input $(x_1, \ldots, x_a)$ (assume $x_1 \sqsubset \cdots \sqsubset x_a$) the only possibilities for $\mathrm{F}^A_a(x_1, \ldots, x_a)$ are 1^i0^{a-i} where $0 \leqslant i \leqslant a$.

By Theorem 5.6 $freq^A_{b,a} \cap \mathrm{SEN}(k(D, a-b)) \neq \emptyset$. Since $0 \leqslant a-b \leqslant a$ we can apply Lemma 5.9 with $r = a-b$. Hence $freq^A_{b,a} \cap \mathrm{SEN}(k) \neq \emptyset$.

Assume, by way of contradiction, that $freq^A_{b,a} \cap \mathrm{SEN}(k-1) \neq \emptyset$. By Theorem 5.6 there exists $\mathscr{D}$ such that A is strongly $\mathscr{D}$-verbose and $k(\mathscr{D}, a-b) = k-1$. By Lemma 5.8 $\#^A_a \in \mathrm{EN}((k-1)(2(a-b)+1)) \subseteq \mathrm{EN}(a)$. By Lemma 2.5 A is recursive. □

5.2. Joins of semirecursive sets

In this section we obtain an upper bound on the complexity of $freq^A_{b,a}$ when A is the join of several semirecursive sets. No lower bound is known in the general case; however, there are particular sets A of this type for which the lower bound is tight.

Joins of semirecursive sets are *not* that interesting; however, they make a nice illustration of the power of our techniques.

Definition 5.11. If D_1 and D_2 are sets of strings then

$$D_1 \cdot D_2 = \{\sigma\tau : \sigma \in D_1 \text{ and } \tau \in D_2\}.$$

Definition 5.12. If $A_1, A_2 \subseteq \mathscr{N}$ then

$$A_1 \oplus A_2 = \{2x : x \in A_1\} \cup \{2x+1 \mid x \in A_2\}.$$

Lemma 5.13. *Let* $a_1, \ldots, a_q$ *and* $D_1, \ldots, D_q$ *be such that* $D_i \subseteq \{0,1\}^{a_i}$ *for all i. Then*

$$k(D_1 \cdot D_2 \cdots D_q, r) \leqslant \min \left\{ \prod_{i=1}^{q} k(D_i, r_i) : (\forall i)[r_i \geqslant 1] \text{ and } \sum_{i=1}^{q} r_i = r \right\}.$$

Proof. We prove this for $q = 2$. The general case is similar. Let $r = r_1 + r_2$ be some partition of r into nonzero parts. Let k_1 and k_2 be such that $k(D_i, r_i) = k_i$. Let $y_1, \ldots, y_{k_1}, z_1, \ldots, z_{k_2}$ be such that $D_1 \subseteq \bigcup_{i=1}^{k_1} B(y_i, r_1)$ and $D_2 \subseteq \bigcup_{i=1}^{k_2} B(z_i, r_2)$. It is easy to see that

$$D_1 \cdot D_2 \subseteq \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{k_2} B(y_i \cdot z_j, r_1 + r_2).$$

Hence $k(D_1 \cdot D_2, r) \leqslant k_1 k_2 = k(D_1, r_1) k(D_2, r_2)$. Since this holds for any nonzero partition $r = r_1 + r_2$ we can take r_1, r_2 that results in the minimal $k(D_1, r_1) k(D_2, r_2)$. □

Theorem 5.14. *Assume* $1 \leqslant b \leqslant a$, $b/a > \frac{1}{2}$, *and* $q \geqslant 1$. *Let* $A_1, \ldots, A_q$ *be semirecursive sets. Let* $A = A_1 \oplus \cdots \oplus A_q$.

1. $freq_{b,a}^A \cap \mathrm{SEN}(k) \neq \emptyset$ *where* k *is defined as follows.*

$$k = \max\left\{ \min\left\{ \prod_{i=1}^{q} \left\lceil \frac{a_i + 1}{2r_i + 1} \right\rceil : \sum_{i=1}^{q} r_i = a - b \right\} : \sum_{j=1}^{q} a_i = a \right\}.$$

2. *If* q *divides both* a *and* b *then* $freq_{b,a}^A \cap \mathrm{SEN}((\lceil (a+q)/(2a-2b+q) \rceil)^q) \neq \emptyset$.

Proof. (1) For any a', $0 \leqslant a' \leqslant a$, let $E^{a'} = \{1^i 0^{a'-i} : 0 \leqslant i \leqslant a'\}$. Note that A is strongly $\mathscr{D}$-verbose where $\mathscr{D} = \{\prod_{i=1}^{q} E^{a_i} : \sum_{j=1}^{q} a_i = a\}$. By Theorem 5.6 $freq_{b,a}^A \cap \mathrm{SEN}(k) \neq \emptyset$ where

$$k = \max\left\{ k\left(\prod_{i=1}^{q} E^{a_i}, a - b \right) : \sum_{j=1}^{q} a_i = a \right\}.$$

By Lemmas 5.13 and 5.9

$$\begin{aligned} k\left(\prod_{i=1}^{q} E^{a_i}, a - b \right) &\leqslant \min\left\{ \prod_{i=1}^{q} k(E^{a_i}, r_i) : \sum_{i=1}^{q} r_i = a - b \right\} \\ &\leqslant \min\left\{ \prod_{i=1}^{q} \left\lceil \frac{a_i + 1}{r_i + 1} \right\rceil : \sum_{i=1}^{q} r_i = a - b \right\}. \end{aligned}$$

Putting this all together we obtain that $freq_{b,a}^A \cap \mathrm{SEN}(k) \neq \emptyset$ where

$$k = \max\left\{ \min\left\{ \prod_{i=1}^{q} \left\lceil \frac{a_i + 1}{2r_i + 1} \right\rceil : \sum_{i=1}^{q} r_i = a - b \right\} : \sum_{j=1}^{q} a_i = a \right\}.$$

(2) If q divides b and a, then q divides $a - b$. In this case the internal min occurs when all r_i's are $(a - b)/q$. Hence,

$$k = \max\left\{ \prod_{i=1}^{q} \left\lceil \frac{a_i + 1}{(2a - 2b)/q + 1} \right\rceil : \sum_{j=1}^{q} a_i = a \right\}.$$

The max occurs when all a_i's are a/q. When this occurs

$$k=\prod_{i=1}^{q}\left\lceil \frac{a/q+1}{(2a-2b)/q+1}\right\rceil = \left(\left\lceil \frac{a/q+1}{(2a-2b)/q+1}\right\rceil\right)^q = \left(\left\lceil \frac{a+q}{2a-2b+q}\right\rceil\right)^q. \quad \square$$

There are semirecursive sets $A_1,\ldots,A_q$ where the upper bound from Theorem 5.14 is an overestimate; for example, if $A_1=\cdots=A_q$ then $\mathrm{F}_a^{A_1\oplus\cdots\oplus A_q}\in \mathrm{SEN}(a+1)$. However, Theorem 5.14 is optimal for the general case.

Theorem 5.15. *Let a,b,q,k be as in Theorem 5.14. There exist sets $A,A_1,\ldots,A_q$ such that $A=A_1\oplus\cdots\oplus A_q$ and $freq_{b,a}^A\cap \mathrm{EN}(k-1)=\emptyset$.*

Proof. This can be proven by a straightforward diagonalization similar to [11, Appendix]. □

5.3. Superterse and weakly superterse sets

Clearly, for all A and n, $\mathrm{F}_n^A\in \mathrm{FQ}(n,A)$. There are sets for which F_n^A *requires n* queries. These sets make F_n^A as hard as possible in terms of queries. The next definition defines such sets rigorously.

Definition 5.16 (Beigel et al. [5]). A set A is *superterse* if $(\forall n)(\forall X)[\mathrm{F}_n^A\notin \mathrm{FQ}(n-1,X)]$. A set A is *weakly superterse* if $(\forall n)(\forall X)[\mathrm{F}_n^A\notin \mathrm{FQC}(n-1,X)]$.

Clearly, for all A and n, $\mathrm{F}_n^A\in \mathrm{EN}(2^n)$. There are sets for which $\mathrm{F}_n^A\notin \mathrm{EN}(2^n-1)$. These sets make F_n^A as hard as possible in terms of enumerability. The next lemma states that these are *exactly* the superterse sets.

Lemma 5.17 (Beigel [4]). *Let $A\subseteq\mathcal{N}$.*

1. *If for some a it holds that $\mathrm{F}_a^A\in \mathrm{EN}(2^a-1)$ ($\mathrm{F}_a^A\in \mathrm{SEN}(2^a-1)$), then there exists a constant c such that $(\overset{\infty}{\forall} n)[\mathrm{F}_n^A\in \mathrm{EN}(n^c)]$ ($\mathrm{F}_n^A\in \mathrm{SEN}(n^c)$).*
2. *Assume A is (weakly) superterse. For all n, $\mathrm{F}_n^A\notin \mathrm{EN}(2^n-1)$ ($\mathrm{F}_n^A\notin \mathrm{SEN}(2^n-1)$). This follows from part 1 and Lemma 2.4.*

(In [4] a complexity-theoretic version of this Lemma 5.17 is proved; however, the proof can be modified to obtain Lemma 5.17.)

If A is superterse then the structure of the set of possibilities for F_n^A is well understood since its just $\{0,1\}^n$. The next theorem uses this structure to obtain tight bounds.

Theorem 5.18. *Assume $1\leqslant b\leqslant a$, $b/a>\frac{1}{2}$, and $A\subseteq\mathcal{N}$.*

1. *$freq_{b,a}^A\cap \mathrm{SEN}(k(a,a-b))\neq\emptyset$. The algorithm that achieves this does not look at the input and runs in constant time.*
2. *If A is superterse then $freq_{b,a}^A\cap \mathrm{EN}(k(a,a-b)-1)=\emptyset$.*
3. *If A is weakly superterse then $freq_{b,a}^A\cap \mathrm{SEN}(k(a,a-b)-1)=\emptyset$.*

Proof. (1) This follows from Theorem 5.6, however we present a simpler proof. Let $k = k(a, a-b)$ and let $p_1, \ldots, p_k$ be the centers of the balls of radius $a-b$ that cover $\{0,1\}^a$. On any input just output an index for the finite set $\{p_1, \ldots, p_k\}$.

(2) Let A be superterse. Assume, by way of contradiction, that $freq^A_{b,a} \cap \mathrm{EN}(k(a,a-b)-1) \neq \emptyset$. By Theorem 5.6 there exists $\mathscr{D}$ such that A is $\mathscr{D}$-verbose and $k(\mathscr{D}, a-b) = k(a,a-b)-1$. Hence, for every $D \in \mathscr{D}$, $k(D,a-b) \leqslant k(a,a-b)-1$ so $|D| \leqslant 2^a - 1$. By Lemma 5.8, $\mathrm{F}^A_a \in \mathrm{EN}(2^a-1)$. By Lemma 5.17, A is not superterse.
(3) Similar to part 2. □

Corollary 5.19. *Assume* $1 \leqslant b \leqslant a$.
1. $freq^K_{b,a} \cap \mathrm{SEN}(k(a,a-b)) \neq \emptyset$ *but* $freq^K_{b,a} \cap \mathrm{SEN}(k(a,a-b)-1) = \emptyset$.
2. *For every nonrecursive set* A, $freq^{A'}_{b,a} \cap \mathrm{SEN}(k(a,a-b)) \neq \emptyset$ *but* $freq^{A'}_{b,a} \cap \mathrm{EN}(k(a,a-b)-1) = \emptyset$. *(Recall that* A' *is the halting problem relative to* A*; see* [21, 24].)
3. *Every nonzero truth-table degree contains a set* A *such that* $freq^A_{b,a} \cap \mathrm{SEN}(k(a,a-b)) \neq \emptyset$ *but* $freq^A_{b,a} \cap \mathrm{EN}(k(a,a-b)-1) = \emptyset$.

Proof. By [11, Theorem 23], K is weakly superterse. By [5, Theorem 16], for all nonrecursive A, A' is superterse. By [5, Theorem 14], every nonzero tt-degree contains a superterse set. □

Theorems 4.1 and Corollary 5.19 offer an interesting contrast. We obtain the exact complexity of $freq^K_{b,a}$ via (1) algorithms that need not halt if a different oracle is used, and (2) algorithms that halt regardless of the oracle. Table 1 shows that the difference in complexity is small when $b \leqslant a/2 + 2$, but is exponentially large when $a-b$ is constant. We show how the table is derived and impose conditions as to when the rows of the table apply. The condition $b \leqslant a$ always applies.

1. If $2b = a+4$ then $a = 2(a-b)+4$, hence $k(a,a-b) = k(2(a-b)+4, a-b)$. If $a-b \geqslant 1$ then, by Fact 5.3, $k(2(a-b)+4, a-b) \in \{7, \ldots, 12\}$; hence, by Corollary 5.19 and Lemma 2.4, the optimal number of queries needed to compute $freq^K_{b,a}$ is $\lceil \log k(a,a-b) \rceil \in \{3,4\}$. This derivation only applies when $a-b \geqslant 1$, hence the first row of the table may be excluded in the case $a \leqslant 4$. Also note that a must be even; hence, the condition can be stated as $a \geqslant 6$ and a even.

2. If $2b = a+3$ then $a = 2(a-b)+3$, hence $k(a,a-b) = k(2(a-b)+3, a-b)$. If $a-b \geqslant 1$ then, by Fact 5.3, $k(2(a-b)+3, a-b) = 3$; hence, by Corollary 5.19 and Lemma 2.4, the optimal number of queries needed to compute $freq^K_{b,a}$ is $\lceil \log k(a,a-b) \rceil = 2$ This derivation only applies when $a-b \geqslant 1$, hence the second row of the table may be excluded in the case $a \leqslant 3$. Also note that a must be odd; hence, the condition can be stated as $a \geqslant 5$ and a odd.

3. If $2b = a+2$ then $a = 2(a-b)+2$, hence $k(a,a-b) = k(2(a-b)+2, a-b)$. If $a-b \geqslant 1$ then, by Fact 5.3, $k(2(a-b)+2, a-b) = 2$; hence, by

Table 1

	FQC complexity	FQ complexity	Conditions
$2b = a + 4$	3 or 4	2	$a \geqslant 6, a$ even
$2b = a + 3$	2	2	$a \geqslant 5, a$ odd
$2b = a + 2$	1	1	$a \geqslant 4, a$ even
$b = a - c$	$a - \Theta(c \log a)$	$\log a - \log c + \Theta(1)$	$c \ll a, b$
$b = a - 1$	$a - \Theta(\log a)$	$\log a + \Theta(1)$	$c \ll a, b$

Corollary 5.19 and Lemma 2.4, the optimal number of queries needed to compute $freq^K_{b,a}$ is $\lceil \log k(a, a-b) \rceil = 1$. This derivation only applies when $a - b \geqslant 1$, hence the third row of the table may be excluded in the case $a \leqslant 2$. Also note that a must be even; hence, the condition can be stated as $a \geqslant 4$ and a odd.

4. If $b = a - c$ then $k(a, a-b) = k(a, c)$. By Corollary 5.19 and Lemma 2.4 the optimal number of queries needed to compute $freq^K_{b,a}$ is $\lceil \log k(a, a-b) \rceil = \lceil \log k(a, c) \rceil$. If $a, b >> c$ then, by Fact 5.3, this is $a - \Theta(c \log a)$.

6. Complexity theory

Several of our results have analogues in complexity theory.

Definition 6.1. Let $X \subseteq \Sigma^*$ and let $k \in \mathcal{N}$. Then $\mathrm{PF}^{X[k]}$ is the set of functions that can be computed in polynomial time with k queries to X. A set $A \subseteq \Sigma^*$ is *p-superterse* if $(\forall k)(\forall X)[\mathrm{F}^A_k \notin \mathrm{PF}^{X[k-1]}]$. A function f is *k-enumerable in polynomial time* if there exists $g \in \mathrm{PF}$ such that $g(x)$ produces k values, one of which is $f(x)$. We denote this by $f \in \mathrm{SEN}(k)$. Note that in this context "strongly k-enumerable" is the same as k-enumerable.

It is easy to see that analogues of Theorems 5.6 and 5.18 hold in a polynomial framework. Applying the analogue of Theorem 5.18 directly is hard since few sets have been shown to be p-superterse outright. However, the following is known [1, 6, 20].

Fact 6.2. *If* $\mathrm{P} \neq \mathrm{NP}$ *then* SAT *is p-superterse.*

Combining Fact 6.2 with the polynomial analogue of Theorem 5.18 yields the following theorem.

Theorem 6.3. *Assume* $1 \leqslant b \leqslant a$ *and* $A \subseteq \Sigma^*$.

1. $freq^A_{b,a} \cap \mathrm{SEN}(k(a, a-b)) \neq \emptyset$. *The algorithm that achieves this does not look at the input and runs in constant time.*
2. *If* $\mathrm{P} \neq \mathrm{NP}$ *then* $freq^{\mathrm{SAT}}_{b,a} \cap \mathrm{SEN}(k(a, a-b) - 1) = \emptyset$.

Acknowledgements

We would like to thank Richard Chang, James Foster, Martin Kummer, Georgia Martin, Nick Reingold, Dan Spielman, Frank Stephan, and two anonymous referees for proofreading and helpful comments.

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