SACRED HEART UNIVERSITY

Sequences of Nested Tetrahedrons

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Abstract

In this thesis, we construct sequences of nested tetrahedrons from a given tetrahedron using an iterative procedure. We are interested in the limiting behavior of such sequences. We will briefly mention the relevant known results on sequences of nested triangles and generalize to a sequence of nested tetrahedrons.

1 Introduction

The topics explored in this thesis are inspired by results proved by Jacobs in [1] on sequences of nested triangles. She began with a triangle and constructed an infinite sequence of smaller triangles inside via an iterative process. The n^{th} triangle's shape was examined along with the location of the limiting points. We now define the terms needed to understand that iterative process and her results.

Definition 1.1. *Triangles which are* ***similar*** *have the same shape in that their corresponding sides are proportional and corresponding angles are equal.*

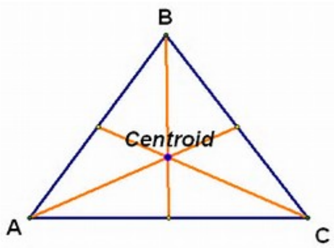

Figure 1: Triangle Centroid

So in proving that the n^{th} triangle in the sequence is similar to the initial triangle, she is saying that every triangle in this sequence looks the same.

Definition 1.2. *The* ***median of the triangle*** *is a line segment from a vertex to the midpoint of the opposite side.*

Definition 1.3. *The* ***centroid of a triangle*** *is the point at which the three medians of the triangle meet. See Figure 1.*

The centroid of a triangle can be represented as $\frac{A+B+C}{3}$ where A, B, and C are the vertices of the triangle. So if we had $\triangle ABC$ with vertices $A = (x_1, y_1)$,

$B = (x_2, y_2)$, and $C = (x_3, y_3)$, then the centroid would be calculated as the point $(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3})$. Now we are ready to formally state the result in [1].

Theorem 1.4 ([1]). *Given $T_0 = \triangle A_0B_0C_0$ an arbitrary triangle, let $T_1 = \triangle A_1B_1C_1$ be a triangle determined by the midpoints of the sides of T_0. Similarly, let $\triangle A_2B_2C_2$ be determined by the midpoints of the sides of $\triangle A_1B_1C_1$. Continuing this process, one gets a sequence of triangles $T_n = \{A_nB_nC_n\}_{n=0}^{\infty}$ which are all similar. Moreover, as n approaches infinity, the sequence converges to a point which is the center of gravity of $\triangle A_0B_0C_0$.*

The goal of this paper is to consider that result from [1] and generalize it to tetrahedrons. A tetrahedron is a solid having four plane triangular faces. We are going to construct our sequence of nested tetrahedrons by taking the centroid of each triangular face and labeling those as the vertices for the next tetrahedron in the sequence. See Figure 2.

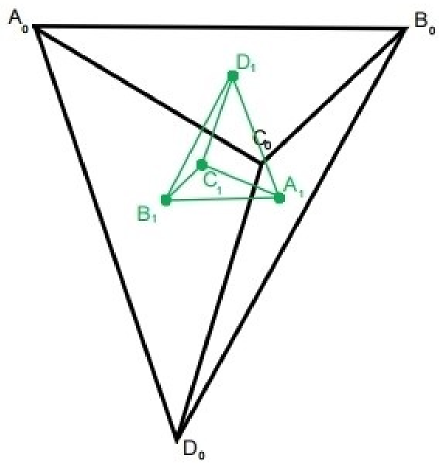

Figure 2: Nested Tetrahedrons

We are going to prove that as n approaches infinity, the n^{th} tetrahedron in our sequence will be of a similar shape to that of the initial tetrahedron and the sequence of nested tetrahedrons will approach the centroid of the initial tetrahedron.

Definition 1.5. *The* ***median of a tetrahedron*** *is the line segment joining a vertex of a tetrahedron with the centroid of the opposite triangular face.*

Definition 1.6. *The* ***bimedian of a tetrahedron*** *is the line segment joining the midpoints of two opposite edges.*

Definition 1.7. *The* ***centroid of a tetrahedron*** *is the point at which the four medians and three bimedians meet. See Figure 3.*

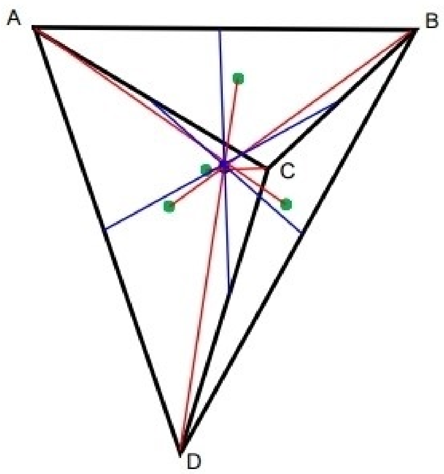

Figure 3: Tetrahedron Centroid

The centroid of a tetrahedron can be represented as $\frac{A+B+C+D}{4}$ where A, B, C, and D are the vertices of the tetrahedron.

2 Review of Linear Algebra and 3 Dimensional Space

Before we get into proving our main theorem, we will review some topics from linear algebra and multivariable calculus. The theorems and definitions reviewed in this section are from [2] and [3]. We direct the reader to these texts for a more detailed explanation of these concepts. The main result that we need is the Diagonalization Theorem, which we state first. Then we will explain the relevant terminology.

Theorem 2.1 (Diagonalization Theorem)**.** *An $n \times n$ matrix M is diagonalizable if and only if M has n linearly independent eigenvectors; and $M = P^{-1}DP$, with D a diagonal matrix, if and only if the rows of P are n linearly independent eigenvectors of M. In this case, the diagonal entries of D are eigenvalues of M that correspond, respectively, to the eigenvectors in P.*

We want to use the Diagonalization Theorem because when we rewrite M as the product of $P^{-1}DP$, we can then use the fact that $M^n = P^{-1}D^nP$. This will allow us to determine what happens to the sequence at its n^{th} term. Then we can take the limit of that product, as n approaches infinity, to conclude to what point the sequence converges.

In order to use the Diagonalization Theorem, we must review the concepts of linear independence, eigenvectors, and eigenvalues.

Definition 2.2. *An indexed set of vectors $\{v_1, ..., v_p\}$ in $\mathbb{R}^n$ is said to be* ***linearly independent*** *if the vector equation*

$$x_1v_1 + x_2v_2 + ... + x_pv_p = 0$$

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has only the trivial solution. (Also the columns of a matrix M are linearly independent if and only if the equation $Mx = 0$ has only the trivial solution).

Definition 2.3. *An* ***eigenvector*** *of an $n \times n$ matrix M is a nonzero vector x such that $Mx = \lambda x$ for some scalar λ.*

Definition 2.4. *A scalar λ is called an* ***eigenvalue*** *of M if there is a nontrivial solution x of $Mx = \lambda x$.*

The way that we will construct the matrix P for the Diagonalization Theorem is by taking the eigenvectors of M as its rows, and we will construct the matrix D containing the eigenvalues of M along its diagonal and zeros everywhere else.

In order to explore the shape similarity of the nested tetrahedrons, we will use three dimensional vectors. Recall from multivariable calculus the concept of finding the cross product of two vectors, defined below.

Definition 2.5. *If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the* ***cross product*** *of a and b is the vector*

$$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

Theorem 2.6. *Two nonzero vectors a and b are parallel if and only if $a \times b = 0$.*

We will find a vector which is on the same plane as one of the triangular faces of the initial tetrahedron and compare that to a vector on the plane of the respective face of the next tetrahedron in the sequence. If the cross product of those two vectors equals zero, then it follows that the planes are parallel. This will be useful in our proof analyzing the limiting shape of the sequence of tetrahedrons.

3 The Centroid Tetrahedron Sequence

Theorem 3.1. *Given* $T_0 = A_0B_0C_0D_0$ *an arbitrary tetrahedron, let* $T_1 = A_1B_1C_1D_1$ *be a tetrahedron determined by connecting the centroids of the triangular faces of* T_0*. Similarly, let* $A_2B_2C_2D_2$ *be a tetrahedron determined by connecting the centroids of the triangular faces of tetrahedron* $A_1B_1C_1D_1$*. Continuing this process, we get a sequence of tetrahedrons* $T_n = \{A_nB_nC_nD_n\}_{n=0}^{\infty}$ *which are all similar. Also, as n approaches infinity, the sequence converges to the centroid of the tetrahedron* $A_0B_0C_0D_0$*.*

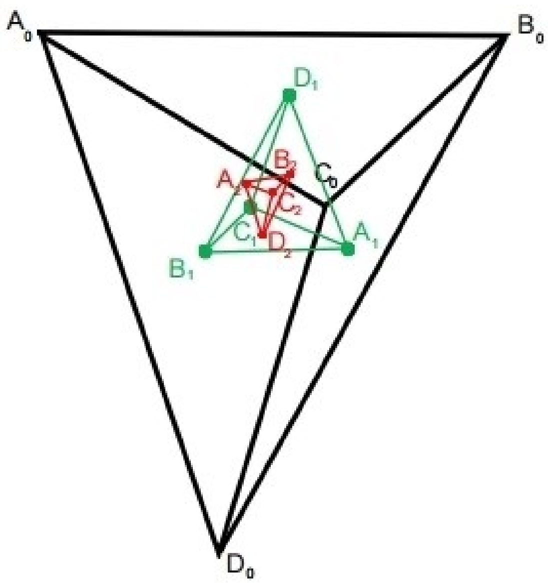

Figure 4: Nested Tetrahedrons

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Proof. Let $A_0B_0C_0D_0$ be any tetrahedron. Construct tetrahedron $A_1B_1C_1D_1$ as stated in Theorem 3.1. Then A_1,B_1,C_1, and D_1 are the centroids of triangular faces $B_0C_0D_0$, $A_0C_0D_0$, $A_0B_0D_0$, and $A_0B_0C_0$ respectively. Let $\overrightarrow{N_0}$ be the normal vector resulting from the cross product of the vectors $\overrightarrow{A_0B_0}$ and $\overrightarrow{A_0C_0}$ and let $\overrightarrow{N_1}$ be the normal vector resulting from the cross product of the vectors $\overrightarrow{B_1A_1}$ and $\overrightarrow{C_1A_1}$. Observe that $\overrightarrow{B_1A_1} = \frac{1}{3}\overrightarrow{A_0B_0}$ and $\overrightarrow{C_1A_1} = \frac{1}{3}\overrightarrow{A_0C_0}$. It follows that $\overrightarrow{N_1} = \frac{1}{9}\overrightarrow{N_0}$ and because they are scalar multiples, $N_0 \times N_1 = 0$. This means that these normal vectors of the triangular faces $A_0B_0C_0$ and $A_1B_1C_1$ are parallel. Thus the planes of $\triangle A_0B_0C_0$ and $\triangle A_1B_1C_1$ are parallel. By induction, it follows that $\triangle A_0B_0C_0$ is parallel to $\triangle A_nB_nC_n$ where n is some positive integer. Since each face of each nested tetrahedron is parallel to the opposite face of the previous one, the angles between the faces of each nested tetrahedron are the same as the original tetrahedron. In other words, $\triangle A_0B_0C_0$ is parallel to $\triangle A_1B_1C_1$, $\triangle A_0B_0D_0$ is parallel to $\triangle A_1B_1D_1$, and $\triangle A_0C_0D_0$ is parallel to $\triangle A_1C_1D_1$. Thus the angle between the planes of $\triangle A_0B_0C_0$ and $\triangle A_0B_0D_0$ is equivalent to the angle between the planes of $\triangle A_1B_1C_1$ and $\triangle A_1B_1D_1$, and the angle between the planes of $\triangle A_0B_0C_0$ and $\triangle A_0C_0D_0$ is equivalent to the angle between the planes of $\triangle A_1B_1C_1$ and $\triangle A_1C_1D_1$. So the angle at vertex A_0 is the same as the angle at vertex A_1 which is equivalent to the angle at the vertex A_n for all n. Therefore, all the nested tetrahedrons in the sequence are similar and the limiting shape is that of the initial tetrahedron.

Now we want to consider the limiting point of our sequence of nested tetrahedrons. Let A_0, B_0, C_0, and D_0 represent their respective coordinates in $\mathbb{R}^3$ as follows: $A_0 = (x_1, y_1, z_1)$, $B_0 = (x_2, y_2, z_2)$, $C_0 = (x_3, y_3, z_3)$, and $D_0 = (x_4, y_4, z_4)$. Then by the definition of the centroid of a triangle,

$$A_1 = \tfrac{B_0+C_0+D_0}{3};$$

$$B_1 = \tfrac{A_0+C_0+D_0}{3};$$

$$C_1 = \tfrac{A_0+B_0+D_0}{3};$$

$$D_1 = \tfrac{A_0+B_0+C_0}{3}.$$

The equations above can also be written in matrix form:

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \\ D_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix}.$$

Using an iterative process, we can calculate the matrix equation for the vertices of each tetrahedron in the sequence:

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \\ D_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}^2 \begin{bmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \\ D_{n-1} \end{bmatrix} = ...$$

$$= \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{bmatrix}.$$

We are interested in M^n as n approaches infinity where

$$M = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}.$$

The matrix M^n will show us to where the vertices of the nested tetrahedrons converge. We can find this behavior by diagonalizing the matrix M using the Diagonalization Theorem.

The eigenvalues of M are $1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}$ and the corresponding eigenvectors are $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}$ respectively. Then

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

$$P^{-1} = \begin{bmatrix} 1/4 & -1/4 & -1/4 & -1/4 \\ 1/4 & 3/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 3/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}.$$

Now we can look at D^n to find M^n:

$$D^n = \begin{bmatrix} 1^n & 0 & 0 & 0 \\ 0 & (-1/3)^n & 0 & 0 \\ 0 & 0 & (-1/3)^n & 0 \\ 0 & 0 & 0 & (-1/3)^n \end{bmatrix}.$$

By taking the limit as n goes to infinity, we get

$$\lim_{n\to\infty} D^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$\lim_{n\to\infty} M^n = \lim_{n\to\infty} P^{-1}D^nP = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

Therefore

$$\lim_{n\to\infty} A_{n+1} = \frac{1}{4}A_0 + \frac{1}{4}B_0 + \frac{1}{4}C_0 + \frac{1}{4}D_0;$$

$$\lim_{n\to\infty} B_{n+1} = \frac{1}{4}A_0 + \frac{1}{4}B_0 + \frac{1}{4}C_0 + \frac{1}{4}D_0;$$

$$\lim_{n\to\infty} C_{n+1} = \frac{1}{4}A_0 + \frac{1}{4}B_0 + \frac{1}{4}C_0 + \frac{1}{4}D_0;$$

$$\lim_{n\to\infty} D_{n+1} = \frac{1}{4}A_0 + \frac{1}{4}B_0 + \frac{1}{4}C_0 + \frac{1}{4}D_0.$$

Thus, all four vertices of the $(n+1)^{th}$ tetrahedron converge to the centroid of tetrahedron $A_0B_0C_0D_0$ as n approaches infinity.

□

4 Conclusion

In this paper, we were able to generalize Theorem 1.4 from [1] to a sequence of nested tetrahedrons, constructed by using the centroids of each face. It is interesting to see that creating a similar sequence in the third dimension, as opposed to the second, will yield similar results. Further explorations can be done by trying to generalize other results in [1]. Different types of nested sequences of triangles were considered in that paper. We hope to recreate those constructions with tetrahedrons.

References

[1] Jacobs, J. (2004). On Sequences of Nested Triangles. Master's thesis, Hofstra University.

[2] Lay, D. C. (2006). *Linear Algebra and its Applications.* Pearson Education, Inc.

[3] Stewart, J. (2012). *Multivariable Calculus.* Brooks/Cole, Cengage Learning.