CONNECTING THE DIGITAL ROOT AND TRIGG OPERATOR

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Abstract

The purpose of this paper will be to look into the digital root of a number and the Trigg Operator and to explore any connections between the two. The Trigg Operator is the operation on the four-digit positive integer n, where $n = abcd$ and $a \geq b \geq c \geq d$ with $a \neq d$ defined as $T(n) = badc - cdab$. The digital root of a number, denoted as dr(n), is the process of repeatedly adding the digits of a number and the resulting sum, until the sum is a single digit. We will use different topics in number theory and abstract algebra to better understand the relationship between these two topics.

1 Introduction

The two elements in this paper, the digital root and the Trigg Operator, are both an area of mathematics called number theory. The digital root is the repeated sum of a positive integer and the resulting integers until a one-digit result is reached, formally defined in Definition 2.1. Charles W. Trigg created the Trigg Operator [3], defining it as the operator on a four-digit $n = abcd$ where $a \geq b \geq c \geq d$ and $a \neq d$ such that $T(n) = badc - cdab$, formally defined in Definition 2.1. We will then look into a connection that exists between these two topics. Finally we will end this paper by seeing how the connection between these two topics can be extended.

2 The Digital Root

We will begin by defining the digital root.

DEFINITION 2.1 The *digital root* of a number is the one-digit number obtained by adding all the digits of the original number to obtain a new number, then adding all of the digits of the new number to obtain a third number, and so on until a one-digit result is obtained [1].

The digital root will be denoted as $dr(n)$ for all $n \in \mathbb{Z}^+$. The specific iterations of the digital root will be denoted as ${}^{i}dr(n)$ where i the number of the iteration.

Additionally n_k is the initial integer where k denotes the number of digits of the positive integer n.

Example: Let's find the digital root of $n_4 = 1234$ or $dr(1234)$. First we must add each digit $1 + 2 + 3 + 4 = 10$ which gives us ${}^1dr(1234) = 10$. This is not the final answer since the result is not one digit. To continue we repeat the process and now find $dr(10)$, so ${}^2dr(1234) = 1 + 0 = 1$. Since the result is a one digit answer we are finished and the digital root of 1234 would be: $dr(1234) = 1$.

Now that we have a basic understanding of the digital root let's look into some of the properties associated with it. We will begin by proving that all positive integers have a single digit result from the digital root.

THEOREM 2.2 *The result of the digital root of any positive integer, n, will be a one-digit number, regardless of the number of digits n has.*

Proof: Let's start with the case that n has one digit, $n_1 = a$. Then the sum of the digit is just a itself, which is a one digit number. Next let us look at the case where n has two digits, $n_2 = a(10) + b$. Then ${}^1dr(n_2)$ can either be a one digit number, if $a + b \leq 9$, or can be another two digit number, if $a + b \geq 10$. If $a + b \geq 10$ then the largest possible case would be if $n_2 = 99$ and ${}^1dr(99) = 18$, ${}^2dr(99) = 1 + 8 = 9$. Any other smaller n_2 will have a ${}^1dr(n_2)$ that is less than 18 and will reduce to a single digit after the second iteration or ${}^2dr(n_2)$. That is true because $dr(n)$ for $n \leq 18$ has a highest possible value of 9 (as all positive integers less than 18 will either be a single digit or will have a sum of one plus a digit less that or equal to eight). Therefore for n_2 it will take a maximum of two iterations for the result to be a single digit. For $k > 2$ the maximum value of n_k is $10^k - 1$ (for example if $k = 4$ then the maximum value of n_4 is 9999 which can be written as $10^4 - 1$). While the maximum value of ${}^1dr(n_k)$ is $9k$. Observe that $10^k - 1$ will always have more digits than $9k$ for all $k > 2$. In other words for $k > 2$, n_k will always have more digits than ${}^1dr(n_k)$. This means that we can then let the result from ${}^1dr(n_k)$ be our new n and continue with iterations of the digital root, each time resulting in an integer with fewer digits until $k = 2$ or $k = 1$. Since k is arbitrary then the digital root of any number, regardless of the number of digits, will have a single digit final result. $\square$

Next we will look at patterns for determining the final result of the digital root. We will start off with the base case of $dr(n)$ for $n = 0$.

LEMMA 2.3 $dr(n) = 0$ *if and only if* $n = 0$ *for* n *in* $\mathbb{Z}^+$.

Proof: There are no positive integers whose sum is 0, therefore if $dr(n) = 0$, then n must be equal to 0, likewise $dr(0)$ is equal to 0. $\square$

Let's look into what $dr(n)$ is for $n \in 9\mathbb{Z}$ for some positive integer j as this will be useful in a later proof.

LEMMA 2.4 *If n is divisible by 9, for some n in $\mathbb{Z}^+$, then $dr(n) = 9$.*

Proof: Let's begin by rewriting n as:

$$n = x_i(10^i) + x_{i-1}(10^{i-1}) + ... + x_0(10^0),\ for\, x_i \neq 0$$

and for some i in $\mathbb{Z}^+$. Then n can also be written as the digits of n multiplied by $[(10^i - 1) + 1]$ for each respective i:

$$n = x_i((10^i - 1) + 1) + x_{i-1}((10^{i-1} - 1) + 1) + ... + x_0((10^0 - 1) + 1).$$

This is equivalent to:

$$n = x_i(10^i - 1) + x_i + x_{i-1}(10^{i-1} - 1) + x_{i-1} + ... + x_0(10^0 - 1) + x_0.$$

Since the each $(10^h - 1)$ term is divisible by 9 then for n to be divisible by 9, it follows that, $x_i + x_{i-1} + ... + x_0$ must be divisible by 9. Let's define $x_i + x_{i-1} + ... + x_0 = m$. Note that $m = {}^1dr(n)$ and m is in $\mathbb{Z}^+$ and is divisible by 9. Thus we can decompose m the same way we did n to find that the sum of the digits of m (which is ${}^2dr(n)$) are divisible by 9. As we continue to find ${}^kdr(n)$ each k^{th} iteration will be divisible by 9. We will eventually have a result that is a single digit ($dr(n)$ as we know from Theorem 2.2), that will be divisible by 9. We assumed that $n \neq 0$, because n is in $\mathbb{Z}^+$, thus the only other single digit that is divisible by 9 is 9 itself. Hence $dr(n)$ must be equal to 9. $\square$

So far we have looked at the $dr(n)$ for $n = 0$ and for $n = 9k$, with some positive integer k. We will classify all other n into one group that follow the same pattern, but first we must review some basic properties of modular arithmetic, which for the purposes of this paper, we will define as the non-negative remainder r left over when we divide some integer by a positive n.

LEMMA 2.5 $(a+b)\ (mod\ n) \ = \ [(a\ (mod\ n)) + (b\ (mod\ n))]\ (mod\ n).$

<u>**Proof**</u>: Let $a = qn + r$ and $b = kn + s$ for some positive integers a, b, n and non-negative integers r, s (by the division algorithm [2]). Notice $a \pmod{n} = r$ and $b \pmod{n} = s$, since $0 \leq r, s < n$ and $a = qn + r$ and $b = kn + s$. Then:

$$\begin{aligned}(a+b) \pmod{n} &= [(qn+r) + (kn+s)] \pmod{n} \\ &= [(q+k)n + r + s] \pmod{n} \\ &= [r+s] \pmod{n} \\ &= [(a \pmod{n}) + (b \pmod{n})] \pmod{n}.\end{aligned}$$ $\square$

LEMMA 2.6 $(ab)\ (mod\ n) \ = \ [(a\ (mod\ n)) \cdot (b\ (mod\ n))]\ (mod\ n).$

Proof: Let $a = qn + r$ and $b = kn + s$ for some positive integers a, b, n and non-negative integers r, s (by the division algorithm [2]). Notice $a \pmod n = r$ and $b \pmod n = s$, since $0 \leq r, s < n$ and $a = qn + r$ and $b = kn + s$. Then:

$$\begin{aligned}
(ab) \pmod n &= ((qn + r)(kn + s)) \pmod n \\
&= [qkn^2 + qns + knr + rs] \pmod n \\
&= [(qkn + qs + kr)n + rs] \pmod n \\
&= [rs] \pmod n \\
&= [(a \pmod n) \cdot (b \pmod n)] \pmod n. \qquad \square
\end{aligned}$$

Now we are sufficiently prepared to prove what the result of the digital root of of any non-negative integer that is not divisible by 9 will be.

THEOREM 2.7 *If $n \neq 0$ and $n \neq 9j$ for some j, n in $\mathbb{Z}^+$ then $dr(n) = n \ (mod\ 9)$.*

Proof: Recall that $(a + b) \bmod 9 = [(a \bmod 9) + (b \bmod 9)] \pmod 9$ by Lemma 2.5. Similarly recall that $(ab) \pmod 9 = [(a \bmod 9) \cdot (b \bmod 9)] \pmod 9$ by Lemma 2.6. So then for $n_k = x_{k-1}(10^{k-1}) + x_{k-2}(10^{k-2}) + ... + x_0(10^0)$, $n_k \pmod 9$ would be equal to:

$$\begin{aligned}
&= (x_{k-1}(10^{k-1}) + x_{k-2}(10^{k-2}) + ... + x_0(10^0) \pmod 9) \\
&= [(x_{k-1}(10^{k-1}) \pmod 9) + x_{k-2}(10^{k-2}) \pmod 9) + ... + x_0(10^0) \pmod 9)] \pmod 9) \\
&= [(x_{k-1} \pmod 9) + x_{k-2} \pmod 9) + ... + x_0 \pmod 9)] \pmod 9)
\end{aligned}$$

$$= [x_{k-1} + x_{k-2} + ... + x_0] \pmod 9$$

$$= {}^1dr(n_k) \pmod 9.$$

Now if we repeat the process from above with ${}^1dr(n_k)$ as our new n_k, we see that the result will be the next iteration of the digital root (mod 9). Thus we find: ${}^1dr(n_k) \pmod 9 = {}^2dr(n_k) \pmod 9 = {}^3dr(n_k) \pmod 9 = ... = {}^idr(n_k) \pmod 9$. This process can be continued until we reach ${}^idr(n_k) \pmod 9$ where ${}^idr(n_k)$ is equal to a one-digit positive integer. So since ${}^idr(n_k) \pmod 9$ is equal to some one-digit integer, h, we know that ${}^idr(n_k) \pmod 9 = h = dr(n_k)$. Therefore it follows that for $n \neq 9j$, $dr(n) = n \pmod 9$. $\square$

If we put together the results proved above in Lemma 2.3, Lemma 2.4, and Theorem 2.7 we have the resulting equation:

$$dr(n_k) = \begin{cases} 0 & \text{if } n_k = 0 \\ 9 & \text{if } n_k = 9j \text{ for some } j \in \mathbb{Z}^+ \\ n_k \pmod 9 & \text{if } n_k \neq 9j \text{ for some } j \in \mathbb{Z}^+ \end{cases} \tag{2.1}$$

3 The Trigg Operator

We will begin by defining the Trigg Operator and the constraints that must be in place when utilizing it.

DEFINITION 3.1 Let n be a four-digit number, defined as $n = a(1000) + b(100) + c(10) + d$ where $9 \geq a \geq b \geq c \geq d \geq 0$ and $a \neq d$. The *Trigg Operator*, $T(n)$, is then defined as:

$$T(n) = [b(1000) + a(100) + d(10) + c] - [c(1000) + d(100) + a(10) + b].$$

Example:

$T(6431) = [4(1000) + 6(100) + 1(10) + 3] - [3(1000) + 1(100) + 6(10) + 4] = 1449$

Before we investigate the relationships that exist between the Trigg Operator and the Digital Root (Definition 2.1), we will prove Lemma 3.2.

LEMMA 3.2 *Let $k = a(10^i) + b(10^{i-1})$ be an integer where $0 \leq a, b \leq 9$, then k can also be written as $k = (a-1)(10^i) + (b+10)(10^{i-1})$.*

Proof: We must show that $a(10^i) + b(10^{i-1}) = (a-1)(10^i) + (b+10)(10^{i-1})$. If we multiply out the right side the result is:

$$\begin{aligned}(a-1)(10^i) + (b+10)(10^{i-1}) &= a(10^i) - 10^i + b(10^{i-1}) + 10^i \\ &= a(10^i) + b(10^{i-1}).\end{aligned}$$

Hence $(a-1)(10^i) + (b+10)(10^{i-1}) = a(10^i) + b(10^{i-1})$. □

We now show that if we take the Trigg Operator of a number n, the digital root of the resulting number is 9.

THEOREM 3.3 *For any four-digit number* $n = abcd$ *where* $9 \geq a \geq b \geq c \geq d \geq 0$ *and* $a \neq d$*,* $dr(T(n)) = 9$*.*

Proof: By Definition 3.1 the Trigg Operator is

$$T(n) = [b(1000) + a(100) + d(10) + c] - [c(1000) + d(100) + a(10) + b]$$

This proof will have two cases, the first is where $b > c$ and the second being where $b = c$. We will begin with the case where $b > c$. Since we know that $b > c$ and $a > d$ we cannot subtract directly, meaning that not all digits on the left are greater than the corresponding digits on the right, but we can use Lemma 3.2 twice to rewrite the Trigg Operator as:

$T(n) = [b(1000) + (a-1)(100) + (d-1+10)(10) + (c+10)] - [c(1000) + d(100) + a(10) + b]$. Simplifying we obtain:

$T(n) = [b(1000)+(a-1)(100)+(d+9)(10)+(c+10)]-[c(1000)+d(100)+a(10)+b]$.

From here we can combine like terms:

$T(n) = (b-c)(1000) + (a-1-d)(100) + (d+9-a)(10) + (c+10-b)$.

We see that $(b-c), (a-1-d), (d+9-a)$, and $(c+10-b)$ must all be single digits. This is due to Definition 3.1, especially the facts that $b \geq c, (a-1) \geq d, (d+9) \geq a$, and $(c+10) > b$ respectively and that $9 \geq a \geq b \geq c \geq d \geq 0$ with $a \neq d$ (since not all digits are equal and a, b, c, and d are all a single digit each). By Definition 2.1, $dr(T(n))$ is equal to the sum of the digits of $T(n)$. The sum of the digits of the resulting Trigg Operator from above are:

$$\begin{aligned}
dr(T(n)) &= (b-c)+(a-1-d)+(d+9-a)+(c+10-b) \\
&= b-c+a-1-d+d+9-a+c+10-b \\
&= a-a+b-b+c-c+d-d-1+9+10 \\
&= -1+9+10 \\
&= 18
\end{aligned}$$

By Definition 2.1 the sum must be repeated until the result is a one-digit integer we must find $dr(18)$ which is just $1+8 = 9$.

Now we will look at the case where $b = c$ but recall $a > d$ is still true, by Definition 3.1. Just as we did in the first case we will now use Lemma 3.2 once to rewrite the Trigg Operator, so that we will be able to subtract directly:

$$T(n) = [b(1000)+(a-1)(100)+(d+10)(10)+(c)]-[c(1000)+d(100)+a(10)+b].$$

From here we combine like terms:

$$T(n) = (b-c)(1000)+(a-1-d)(100)+(d+10-a)(10)+(c-b).$$

We see that $b-c$ and $c-b$ are both equal to 0 as we have assumed $c = b$. Also we see that $(a-1-d), (d+10-a)$ must be a single digit each, since $a > d$ and both a and d are single digits. Thus we can use use the digital root on this resulting Trigg Operator to obtain:

$$\begin{aligned}
dr(T(n)) &= (b-c)+(a-1-d)+(d+10-a)+(c-b) \\
&= 0+(a-1-d)+(d+10-a)+0
\end{aligned}$$

$$= a - a + d - d - 1 + 10$$

$$= -1 + 10$$

$$= 9$$

Since by Definition 3.1 the only possible cases are that $b > c$ or $b = c$ and we used arbitrary digits for a, b, c, d we can conclude that this will be the result regardless of the digits chosen and that it must be true for all $T(n)$. □

Now that we have proved Theorem 3.3 we will see if we can extend the results beyond the Trigg Operator.

4 Extension of the Trigg Operator

We will begin by defining an operator for a six-digit number, n.

DEFINITION 4.1 Let n be a six-digit number, defined as:

$$n = a(10^5) + b(10^4) + c(10^3) + d(10^2) + e(10) + f.$$

Where $9 \geq a \geq b \geq c \geq d \geq e \geq f \geq 0$ and $a \neq f$. The *Munday Operator*, $M(n)$, is then defined as:

$$M(n) = b(10^5) + a(10^4) + d(10^3) + c(10^2) + f(10) + e-$$
$$[e(10^5) + f(10^4) + c(10^3) + d(10^2) + a(10) + b].$$

Now we will show an example of the Munday Operator.

Example:

$M(986654) = [8(100000)+9(10000)+6(1000)+6(100)+4(10)+5]-[5(100000)+4(10000)+6(1000)+6(100)+9(10)+8] = 349947$

We will now see that the result found in Theorem 3.3 hold for the Munday Operator as well, specifically that the digital root of $M(n)$ will be 9.

THEOREM 4.2 *For any six-digit number $n = abcdef$ where $9 \geq a \geq b \geq c \geq d \geq e \geq f \geq 0$ and $a \neq f$, $dr(M(n)) = 9$.*

Proof: Similar to Theorem 3.3 we will separate this proof into three different cases. With the cases being where $c = d$ but $b \neq e$, $b = c = d = e$, and $c \neq d$. We will begin by rewriting the Munday Operator based on Definition 4.1 as:

$$M(n) = (b-e)(10^5)+(a-f)(10^4)+(d-c)(10^3)+(c-d)(10^2)+(f-a)(10)+(e-b). \tag{4.1}$$

Then for $c = d$ and $b \neq e$ we can rewrite the Munday Operator 4.1, from above, using Lemma 3.2 as:

$$M(n) = (b-e)(10^5) + (a-1-f)(10^4) + (d-1+10-c)(10^3) + (c-1+10-d)(10^2) + (f-1+10-a)(10) + (e+10-b).$$

With the facts from our restriction of $c = d$ but $b \neq e$ and from Definition 4.1

$a > f$, (in addition to the fact that each variable is a single digit to begin with) we know $(b-e), (a-1-f), (d-1+10-c), (c-1+10-d), (f-1+10-a), (e+10-b)$ must all be single digits. Thus we can find the digital root of this resulting Munday Operator to obtain:

$$\begin{aligned} dr(M(n)) &= (b-e)+(a-1-f)+(d-1+10-c)+(c-1+10-d)+ \\ &\quad (f-1+10-a)+(e+10-b) \\ &= (b-e)+(a-1-f)+(d+9-c)+(c+9-d)+(f+9-a)+(e+10-b) \\ &= a-a+b-b+c-c+d-d+e-e+f-f-1+9+9+9+10 \\ &= -1+9+9+9+10 \\ &= 36 \end{aligned}$$

Since 36 is not a one-digit number we must add its digits to complete the digital root, giving us the final answer of $dr(M(n)) = 3+6 = 9$.

In the case where we have $b = c = d = e$ the we can use Lemma 3.2 to write the Munday Operator 4.1, from above, as:

$$\begin{aligned} M(n) &= (b-e)(10^5)+(a-1-f)(10^4)+(d-1+10-c)(10^3)+ \\ &\quad (c-1+10-d)(10^2)+(f+10-a)(10)+(e-b). \end{aligned}$$

With the facts from our restriction of $b = c = d = e$ and from Definition 4.1 $a > f$, (in addition to the fact that each variable is a single digit to begin with) we know $(b-e), (a-1-f), (d-1+10-c), (c-1+10-d), (f+10-a), (e-b)$ must all be single digits. Thus we can find the digital root of this resulting Munday

Operator to obtain:

$$\begin{aligned}
dr(M(n)) &= (b-e)+(a-1-f)+(d-1+10-c)+(c-1+10-d)+ \\
&\quad (f+10-a)+(e-b) \\
&= (b-e)+(a-1-f)+(d+9-c)+(c+9-d)+(f+10-a)+(e-b) \\
&= a-a+b-b+c-c+d-d+e-e+f-f-1+9+9+10 \\
&= -1+9+9+10 \\
&= 27
\end{aligned}$$

Since 27 is not a one-digit number we must add its digits to complete the digital root, giving us the final answer of $dr(M(n)) = 2+7 = 9$.

In the case where we have $c \neq d$ the we can use Lemma 3.2 to write the Munday Operator 4.1, from above, as:

$$\begin{aligned}
M(n) = {} & (b-e)(10^5)+(a-f)(10^4)+(d-1-c)(10^3)+ \\
& (c-1+10-d)(10^2)+(f-1+10-a)(10)+(e+10-b).
\end{aligned}$$

With the facts from our restriction of $c \neq d$ and from Definition 4.1 $a > f$, (in addition to the fact that each variable is a single digit to begin with) we know $(b-e), (a-f), (d-1-c), (c-1+10-d), (f-1+10-a), (e+10-b)$ must all be single digits. Thus we can find the digital root of this resulting Munday Operator to obtain:

$$\begin{aligned}
dr(M(n)) &= (b-e)+(a-f)+(d-1-c)+(c-1+10-d)+ \\
&\quad (f-1+10-a)+(e+10-b)
\end{aligned}$$

$$\begin{aligned}
&= (b-e)+(a-f)+(d-1-c)+(c+9-d)+(f-9-a)+(e+10-b) \\
&= a-a+b-b+c-c+d-d+e-e+f-f-1+9+9+10 \\
&= -1+9+9+10 \\
&= 27
\end{aligned}$$

Since 27 is not a one-digit number we must add its digits to complete the digital root, giving us the final answer of $dr(M(n)) = 2 + 7 = 9$.

Since by Definition 4.1 the only possible cases are that $c = d$ but $b \neq e$, $b = c = d = e$, or $c \neq d$ and we used arbitrary digits for a, b, c, d, e, f we can conclude that this will be the result regardless of the digits chosen and that it must be true for all $M(n)$, hence $dr(M(n)) = 9$. $\square$

5 Conclusion and Further Research

In this paper by exploring the digital root and Trigg Operator we created a piecewise formula that describes the digital root of all non-negative integers (Equation 2.1). Then we were able to connect them by proving that the digital root of the Trigg Operator of any positive four-digit number of the form $n = abcd$ where $a \geq b \geq c \geq d$ and $a \neq d$ is 9 (Theorem 3.3). We then were able to take what we learned from the Trigg Operator and how it interacts with the digital root and create a operator for six digits that still interacted with the digital root in an analogues way (Theorem 4.2).

An area of further research could be to explore Lemma 3.2 and its connection with the results of Theorem 3.3 and Theorem 4.2. Since we were able to recreate our results found in Theorem 3.3 with the Munday Operator, both times using Lemma 3.2, I conjecture that the result is due to the Lemma itself. It seems possible since both operators rearrange their digits and subtract in a way uses Lemma 3.2 then when we take the digital root all the digits cancel, due to the subtraction. What is then leftover is a multiple of 9 depending on how many times the Lemma 3.2 was used, since by the Lemma we would subtract one from one digit and add ten to the next (since $-1 + 10 = 9$). Moreover since we proved in Lemma 2.4 that any integer that is divisible by 9 has a digital root of 9 that is why both of these operator were proven to have a digital root of 9.

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