The Dynamics of the Logistic Map and Difference Equations

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Abstract

Difference equations describe the evolution of a quantity or population whose changes are measured over discrete time intervals. In this paper, we will investigate these types of recursive relations and classify the local stability of their equilibrium points and periodic solutions. In particular, we will examine the dynamics of the logistic map and the long-term behavior that occurs when we modify its parameter.

1 Introduction to Difference Equations

Difference equations are recursive relations that describe the evolution of a quantity or population whose changes are measured over discrete time intervals. These types of systems are different from *differential equations*, which model quantities that change over continuous time intervals. We can denote a first-order difference equation by saying $x_{n+1} = f(x_n)$, which means that the next term of the sequence is a function of the previous term. In other words, the output that we obtain from a difference equation will become our input when we calculate the next term of the recursion.

One variable that we can define for all difference equations is an *initial condition*, which is one or more values that we set to determine subsequent terms of our recursion. Once we specify the initial condition(s) of our difference equation, we can compile the terms of the sequence into a set such that $\{x_0, x_1, x_2, ...\}$, where x_0 is our initial condition. We refer to this set of values as a *unique solution* of the difference equation, because the succeeding terms are evaluated from the initial condition(s) that we specified for our difference equation. If we want to find the n-th term of our unique solution, we would compose the difference equation with itself n times, which means we would calculate $f^n(x_0)$, or the n-th *iteration* of $f(x_0)$. For example, if we want to find the fourth term of the sequence, then we need to compose the difference equation with itself four times. Mathematically, we can denote this computation as $f(f(f(f(x_0)))) = f^4(x_0)$, where x_0 is the initial condition of our difference equation. For more background information on elementary difference equations, see [2].

We can classify difference equations based on their *order*. If we want to define a difference equation of order k, there must be k initial conditions that affect the output of the recursive sequence. For instance, first-order difference equations contain one initial condition that affects the outcome of the next term. A linear map of the form $x_{n+1} = ax_n$ is a first-order difference equation because the terms are determined by a single initial condition and some constant a. One popular example of a first-order difference equation is the logistic map, which has the equation $x_{n+1} = \mu x_n(1 - x_n)$, where μ is a constant greater than zero and $x_0 \geq 0$. Similar to the linear maps above, the next term of the sequence only depends on one initial condition and some constant μ.

Second-order equations involve two initial conditions that affect the output of the sequence. One example is the Fibonacci Recurrence, which has the form $x_{n+1} = x_n + x_{n-1}$. In this expression, the terms of the Fibonacci Recurrence are impacted by two initial conditions, $x_{-1} = 1$ and $x_0 = 1$. We add these values to obtain $x_1 = x_0 + x_{-1} = 2$. Then we add the values $x_1 = 2$ and $x_0 = 1$ to find that $x_2 = 3$. This pattern continues indefinitely to create the Fibonacci Sequence.

Once we determine the order, we can evaluate the *equilibrium points*, or *fixed* *points*, of a difference equation. Equilibrium points are points that get mapped to themselves, and they are denoted as $\overline{x}$. Equilibrium points can be calculated by solving for $\overline{x}$ in the equation $f(\overline{x}) = \overline{x}$. Graphically, equilibrium solutions can be evaluated by finding the intersection(s) of the function defining the difference equation and the identity line, $y = x$ [2]. If we set our initial condition equal to an equilibrium point, we will stay at that same point for infinitely many iterations of the function. In other words, when an equilibrium point acts as our initial condition, the output will always be the same as our input.

When we find the equilibrium points of our difference equation, we can move into the study of stability analysis, which explores the behavior of solutions when the initial condition is close to the difference equation's equilibrium point(s). If the initial term is close to an equilibrium point and the solution converges towards the equilibrium point, then the equilibrium point is considered a *sink*, or *locally asymptotically stable* [2]. Similarly, if the initial value is close to an equilibrium point and the solution moves away from it, then that equilibrium point is a *source*, or *unstable* [2]. We can analyze the stability of equilibria with the quantity $|f'(\overline{x})|$. If $|f'(\overline{x})| < 1$, then $\overline{x}$ is locally asymptotically stable. On the other hand, if $|f'(\overline{x})| > 1$, then $\overline{x}$ is unstable.

From both cases, we can identify the stability of a difference equation when $|f'(\overline{x})| \neq 1$. For these cases, we are working with *hyperbolic difference equations*.

However, we notice a different pattern when $|f'(\overline{x})| = 1$, and we call these expressions *nonhyperbolic difference equations* [2]. For the latter case, when $|f'(\overline{x})| = 1$, we expand into the concept of *semistability*, where an equilibrium point can act as both a sink and a source. For example, in Figure 1, consider the difference equation $x_{n+1} = x_n(x_n^2 - 2x_n + 2)$. If we solve for $\overline{x}$ in the equation $\overline{x} = \overline{x}(\overline{x}^2 - 2\overline{x} + 2)$, we find that the equilibrium points are $\overline{x} = 0$ and $\overline{x} = 1$. When we choose an initial condition such that $0 < x_0 < 1$, we will approach the equilibrium point $\overline{x} = 1$. This is portrayed by the dotted red line in Figure 1, where $x_0 = 0.1$. However, if we choose an initial condition such that $x_0 > 1$, the sequence will diverge to infinity. This movement is shown by the dotted blue line in Figure 1, where $x_0 = 1.1$. Since the difference equation converges to the equilibrium when $x_0 < \overline{x}$ and moves away from the equilibrium point when $x_0 > \overline{x}$ the equation is *semistable from below* [2]. Conversely, if the terms of the sequence converge to an equilibrium when $x_0 > \overline{x}$ and move away from the equilibrium point when $x_0 < \overline{x}$, then the expression would be *semistable from above* [2].

To expand on the notion of equilibria, we can also find *periodic solutions*. So far, we calculated the equilibrium points, which are referred to as period-1 solutions, since we used one iteration of the difference equation to calculate them. When we compose the difference equation with itself, we get the second iteration of the difference equation, denoted as $f^2(x) = f\left(f(x)\right)$. By solving for $\overline{x}$ in the

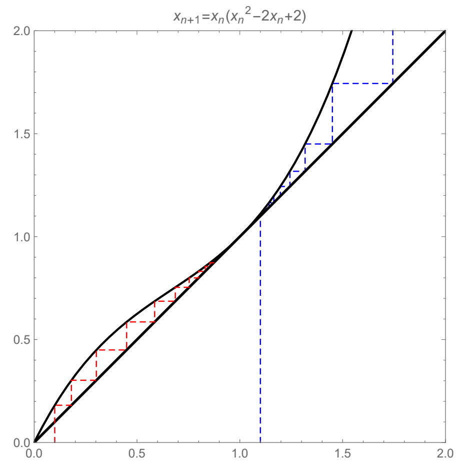

Figure 1: Example of a difference equation that is semistable from below. Here, the equation is $x_{n+1} = x_n(x_n^2 - 2x_n + 2)$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

expression $f^2(\overline{x}) = \overline{x}$, we can find the period-2 solutions of a difference equation. To generalize this idea, we can search for period-n solutions by performing n iterations of the difference equation and attempting to solve for $\overline{x}$ in the equation $f^n(\overline{x}) = \overline{x}$, which may or may not have solutions.

Additionally, when we perform several iterations of our difference equation and calculate higher periodic solutions, we obtain new values and numbers from smaller periods. We call the newfound values the *minimal periodic solutions* of our difference equation. If we set our initial condition x_0 to be one of the minimal period-2 points, the next term of the recursion will be the other minimal period-2 point, and the following term will go back to the first minimal period-2 point. This operation can be expressed mathematically as $x_1 = f(x_0)$ and $x_2 = f(x_1) = x_0$.

The sequence will continue this pattern forever, regardless of how many terms we generate. Additionally, if the minimal period-n solutions are compiled into a set, then we can form a *minimal period-n orbit* of the difference equation. If we look at the above case, the minimal period-2 orbit is $\{x_0, x_1\}$. In general, when we search for the period-n solutions, we will find the minimal period-p solutions such that p divides n, which we denote as $p|n$.

For example, if we compute the fourth iteration of a difference equation and solve for $\overline{x}$ in the expression $f^4(\overline{x}) = \overline{x}$, we will obtain the period-4 solutions. These results consist of the minimal period-4, minimal period-2, and minimal period-1 solutions because $4|4$, $2|4$, and $1|4$, respectively. When we evaluate the period-4 solutions, there will be new values in addition to the minimal period-1 solutions, which are the equilibrium points, and the minimal period-2 solutions. These new quantities are known as the minimal period-4 solutions since those values did not appear when we determined the period-2 and period-1 solutions. If we want to find the period-3 solutions for a difference equation, we get the minimal period-3 solutions and the minimal period-1 values, because $3|3$ and $1|3$. However, we do not acquire the minimal period-2 points, because $2 \nmid 3$.

To showcase periodic and minimal periodic solutions, we will use the tent map to highlight the equilibrium points, period-2, -3, and -4 solutions. The tent map is defined by the difference equation $x_{n+1} = T(x_n)$, where

$$T(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2-2x, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

From there, we can find the equilibrium points by using the equation $T(\overline{x}) = \overline{x}$. Solving for $\overline{x}$, we find that $\overline{x} = 0$ and $\overline{x} = \frac{2}{3}$. The results are shown in Figure 2, where the dotted line represents the identity line $y = x$, and the blue line displays the tent map.

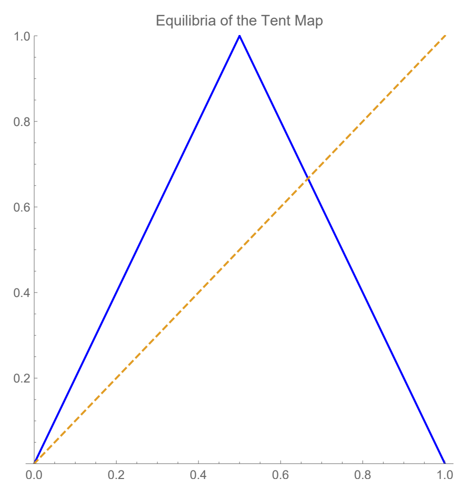

Figure 2: Graph of the tent map $T(x_n)$ and the identity line $y = x$. The minimal period-1 orbit is $\left\{0, \frac{2}{3}\right\}$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

We can take the second iteration of the tent map by composing the function with itself two times. By doing so, we have

$$T^2(x) = \begin{cases} 4x, & \text{if } 0 \le x \le \frac{1}{4} \\ 2-4x, & \text{if } \frac{1}{4} \le x \le \frac{1}{2} \\ 4x-2, & \text{if } \frac{1}{2} \le x \le \frac{3}{4} \\ 4-4x, & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$$

Then we solve for the period-2 solutions with the equation $T^2(\overline{x}) = \overline{x}$ and obtain four values for $\overline{x}$: 0, $\frac{2}{3}$, $\frac{2}{5}$, and $\frac{4}{5}$. From Figure 3, we can see that the second iteration of the tent map intersects the identity line four times. Two of these four points are the equilibrium points of 0 and $\frac{2}{3}$. The other two values of $\frac{2}{5}$ and $\frac{4}{5}$ are the minimal period-2 solutions, because these are new quantities we obtained from the second iteration of the tent map. Thus, our minimal period-2 orbit will be $\left\{\frac{2}{5}, \frac{4}{5}\right\}$.

Likewise, if we solve $T^3(\overline{x}) = \overline{x}$, we will obtain the minimal period-3 solutions $\frac{2}{9}$, $\frac{4}{9}$, $\frac{8}{9}$, $\frac{2}{7}$, $\frac{4}{7}$, and $\frac{6}{7}$, as well as the equilibrium values 0 and $\frac{2}{3}$. If we plug in the minimal period-3 solutions into the original recursion for the tent map, we will find that we obtain two minimal period-3 orbits, $\left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$ and $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$. The third iteration of the tent map and the identity line are displayed in Figure 4.

After calculating the fourth iteration of the tent map, we can find the minimal period-4 solutions $\frac{2}{17}$, $\frac{4}{17}$, $\frac{8}{17}$, $\frac{16}{17}$, $\frac{2}{15}$, $\frac{4}{15}$, $\frac{8}{15}$, $\frac{14}{15}$, $\frac{6}{17}$, $\frac{12}{17}$, $\frac{10}{17}$, and $\frac{14}{17}$. We also obtain the minimal period-2 solutions $\frac{2}{5}$ and $\frac{4}{5}$, and the equilibrium values 0 and $\frac{2}{3}$. If we plug in the minimal period-4 solutions into the original equation for the tent map, we

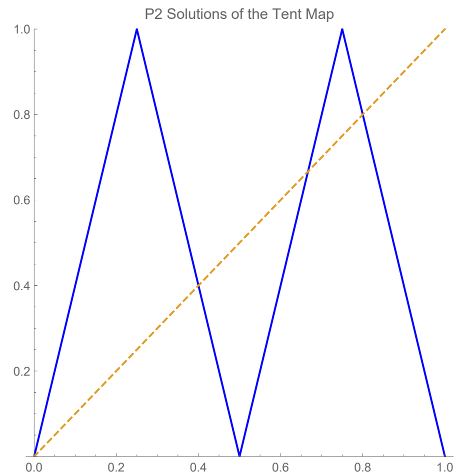

Figure 3: Graph of the second iteration of the tent map $T^2(x)$ and its period-2 solutions, where the minimal period-2 orbit is $\left\{\frac{2}{5}, \frac{4}{5}\right\}$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

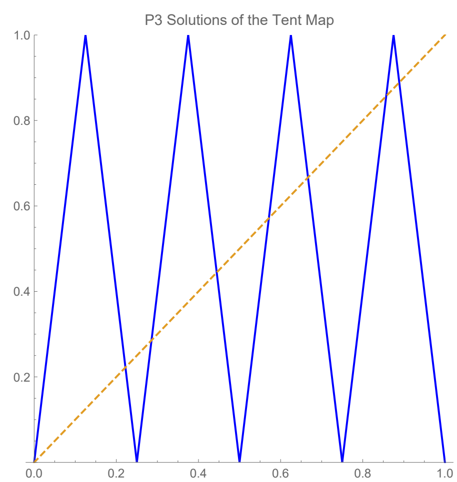

Figure 4: Graph of the third iteration of the tent map $T^3(x)$ and its period-3 solutions, where the minimal period-3 orbits are $\left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$ and $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

find that there are three minimal period-4 orbits, $\left\{\frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{16}{17}\right\}$, $\left\{\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{14}{15}\right\}$, and $\left\{\frac{6}{17}, \frac{12}{17}, \frac{10}{17}, \frac{14}{17}\right\}$. We represent the shape of the fourth iteration of the tent map and its period-4 solutions in Figure 5.

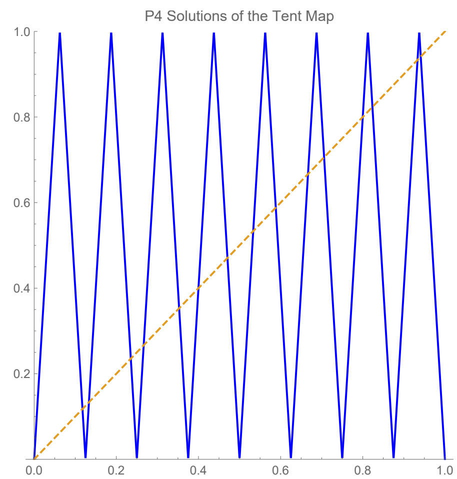

Figure 5: Graph of the fourth iteration of the tent map $T^4(x)$ and its period-4 solutions, where the minimal period-4 orbits are $\left\{\frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{16}{17}\right\}$, $\left\{\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{14}{15}\right\}$, and $\left\{\frac{6}{17}, \frac{12}{17}, \frac{10}{17}, \frac{14}{17}\right\}$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

Another useful component of difference equations is the *basin of attraction*. We define the basin of attraction as the set of initial conditions that will ultimately converge to our equilibrium point(s). This calculation examines which values will approach an equilibrium point, as we keep iterating the difference equation with itself. Mathematically, the basin of attraction is denoted as $\mathcal{B}(\overline{x})$ and can be written as $\mathcal{B}(\overline{x}) = \left\{x \in D : \lim_{n\to\infty} f^n(x) = \overline{x}\right\}$, where D is the domain of the function $f(x)$ [2].

2 Attributes of the Logistic Map:

One of the most popular first-order difference equations is the logistic map. When we discuss the logistic map, we can identify several patterns in the sequence based on the value of its parameter. The logistic map is defined by the recursion $x_{n+1} = \mu x_n(1 - x_n)$, where μ is a constant greater than zero, and $x_0 \geq 0$. For this analysis, we will restrict the domain to the closed interval $[0, 1]$, meaning that $0 \leq x_0 \leq 1$. The equilibrium points can be solved with the formula $f(\overline{x}) = \overline{x}$, which gives us the equation $\overline{x} = \mu\overline{x}(1 - \overline{x})$. Solving for $\overline{x}$, we find that the equilibrium points are $\overline{x} = 0$ and $\overline{x} = 1 - \frac{1}{\mu} = \frac{\mu-1}{\mu}$, where $\mu > 1$. We will refer to $\overline{x} = 0$ as the zero equilibrium, and $\overline{x} = \frac{\mu-1}{\mu}$ as the positive equilibrium.

Now that we established where the equilibrium points are located, we can examine what happens when we modify the constant, μ, and the initial value of the sequence. When $0 < \mu \leq 1$, we will only have one equilibrium point, $\overline{x} = 0$. Moreover, the recursive sequence will converge to $\overline{x} = 0$. We show an example of this case in Figure 6. In this model, the curve represents the logistic map and the identity line of $y = x$. The dotted red line indicates where the recursion begins, and the long-term behavior of its terms. Note that the dotted red line moves back and forth between the logistic map and the identity line because the output that we receive for one term becomes the input when we calculate the next term of the recursion.

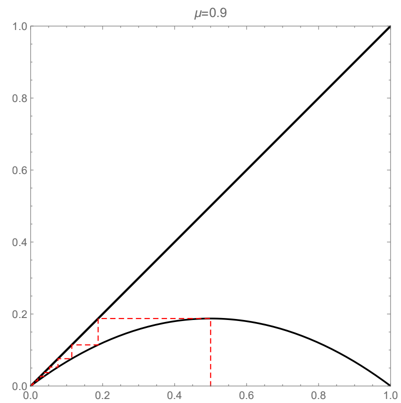

Figure 6: Graph of the logistic map with $0 < \mu \leq 1$. In this case, we have $x_{n+1} = 0.9x_n(1-x_n)$, where $\mu = 0.9$ and $x_0 = 0.5$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

When $1 < \mu < 3$, the sequence will converge to the positive equilibrium, and diverge from the zero equilibrium. We show an example of this scenario in Figure 7. From this case, we can establish the following theorem:

THEOREM 2.1 *For the logistic map* $x_{n+1} = \mu x_n(1 - x_n)$, *when* $1 < \mu < 3$, $\mathcal{B}(\overline{x}) = (0, 1)$, *where* $\overline{x} = \frac{\mu-1}{\mu}$.

The framework for this proof is provided in [3].

Proof: Let $F_\mu(x)$ be the logistic map expressed as a function of x. Thus, $F_\mu(x) = \mu x(1 - x)$. Then $|F'_\mu(x)| = |\mu - 2\mu x|$.

Observe that $F'_\mu(x) = |\mu - 2\mu x| < 1$ if and only if $-1 < \mu - 2\mu x < 1$. Solving for x, we have $\frac{\mu-1}{2\mu} < x < \frac{\mu+1}{2\mu}$.

Therefore, for all $x \in \left(\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right)$, $|F'_\mu(x)| < 1$. Henceforth, we will refer to the interval $\left(\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right)$ as I. Recall that $\overline{x} = \frac{\mu-1}{\mu}$ is the general formula for the positive equilibrium point of the logistic map. We can apply this equation to our inequality from above, which yields $\frac{\mu-1}{2\mu} < \overline{x} < \frac{\mu+1}{2\mu}$. Solving for μ on each side of the inequality, we find that this expression is only satisfied when $1 < \mu < 3$. Consequently, $\overline{x} = \frac{\mu-1}{\mu} \in I$ if and only if $1 < \mu < 3$.

Observe that if we plug the endpoints of interval I into the equation for the logistic map, we find that $F_\mu\left(\frac{\mu-1}{2\mu}\right) = F_\mu\left(\frac{\mu+1}{2\mu}\right) = \frac{\mu^2-1}{4\mu}$. We want the terms of our sequence to lie within this interval since, by the Mean Value Theorem, we find that $|x_n - \overline{x}| = |f(x_{n-1}) - f(\overline{x})| = |f'(c)||x_{n-1} - \overline{x}|$, where $x_{n-1} < c < \overline{x}$ or $\overline{x} < c < x_{n-1}$. Since $|f'(c)| < 1$, it follows that $|x_n - \overline{x}| < |x_{n-1} - \overline{x}|$, which inductively indicates that we are moving closer to the positive equilibrium when we are in the interval I. If we consider the inequality $\frac{\mu-1}{2\mu} < \frac{\mu^2-1}{4\mu} < \frac{\mu+1}{2\mu}$, then we can see that this inequality holds when $1 < \mu < 3$. Thus, $\frac{\mu^2-1}{4\mu} \in I$. Hence, $\left[\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right] \subset \mathcal{B}(\overline{x})$.

We will now consider the interval $\left(0, \frac{\mu-1}{2\mu}\right)$. Let $z \in \left(0, \frac{\mu-1}{2\mu}\right)$. Because $|F_\mu'(x)| < 1$ if and only if $x \in I$, it follows that $F_\mu'(z) > 1$. If we apply the Mean Value Theorem over the interval $(0, z)$, then we have $\frac{F_\mu(z)-F_\mu(0)}{z-0} = F_\mu'(\gamma)$, where $0 < \gamma < z$. Thus, $\frac{F_\mu(z)-F_\mu(0)}{z-0} = \frac{F_\mu(z)}{z} = F_\mu'(\gamma)$. Since $F_\mu'(z) > 1$, it follows that $F_\mu(z) = zF_\mu'(\gamma) \geq \beta z$, for some $\beta > 1$. Then for some $r \in \mathbb{Z}^+$, $F_\mu^r(z) \geq \beta^r z > \frac{\mu-1}{2\mu}$. Additionally, since F is increasing on $\left[0, \frac{\mu-1}{2\mu}\right]$, it follows that

$F_\mu^r(z) = F_\mu(F_\mu^{r-1}(z)) < F_\mu\left(\frac{\mu-1}{2\mu}\right) = \frac{\mu^2-1}{4\mu} \leq \overline{x}$. Hence, $z \in \left(0, \frac{\mu-1}{2\mu}\right) \subset \mathcal{B}(\overline{x})$.

Lastly, we will consider the interval $\left(\frac{\mu+1}{2\mu}, 1\right)$. Let $p \in \left(\frac{\mu+1}{2\mu}, 1\right)$. Note that over the interval $\left[\frac{\mu+1}{2\mu}, 1\right]$, F is decreasing, and $F_\mu(1) = 0$. Thus, $0 = F_\mu(1) < F_\mu(p) < F_\mu\left(\frac{\mu+1}{2\mu}\right) = F_\mu\left(\frac{\mu-1}{2\mu}\right) = \frac{\mu^2-1}{4\mu} \leq \overline{x}$.

Thus, $F_\mu\left(\frac{\mu+1}{2\mu}\right) \subset (0, \overline{x})$, which implies that $p \in \left(0, \frac{\mu+1}{2\mu}\right) \subset \mathcal{B}(\overline{x})$.

Therefore, since $\left(0, \frac{\mu-1}{2\mu}\right) \subset \mathcal{B}(\overline{x})$, $\left[\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right] \subset \mathcal{B}(\overline{x})$, and $\left(\frac{\mu+1}{2\mu}, 1\right) \subset \mathcal{B}(\overline{x})$, it follows that $\mathcal{B}(\overline{x}) = (0, 1)$. $\square$

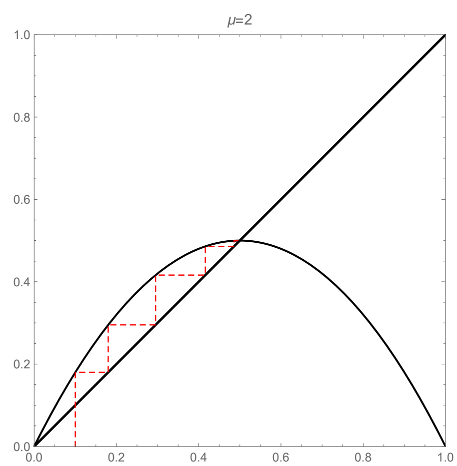

Figure 7: Graph of the logistic map with $1 < \mu < 3$. In this case, we have $x_{n+1} = 2x_n(1 - x_n)$, where $\mu = 2$, $x_0 = 0.1$, and the positive equilibrium $\overline{x} = \frac{1}{2}$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

When $3 < \mu < 1 + \sqrt{6}$, the non-equilibrium solutions converge to the minimal period-2 solutions of the logistic map. Figure 8 showcases this movement towards the minimal period-2 solutions.

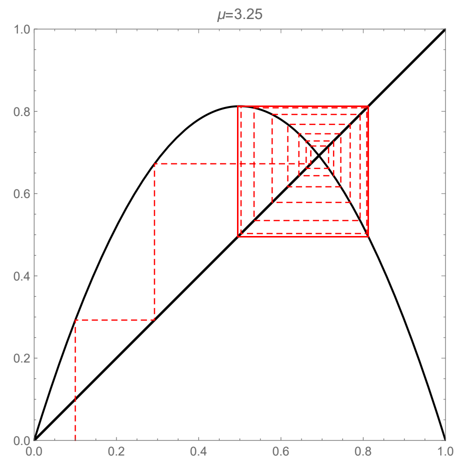

Figure 8: Graph of the logistic map with $3 < \mu < 1 + \sqrt{6}$. In this case, we have $x_{n+1} = 3.25x_n(1 - x_n)$, where $\mu = 3.25$ and $x_0 = 0.1$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

We see that for certain values of μ such as 1, 3, and $1 + \sqrt{6}$, the behavior of the logistic map changes. These three quantities are called *bifurcation values* of the logistic map. Bifurcation values result from a qualitative change in the behavior of the recursion. These numbers can be visualized within a bifurcation diagram, as shown in Figure 9. This illustration measures the quantity of μ on the horizontal axis and the value(s) of the equilibrium point(s) $\overline{x}$ on the vertical axis.

If we take values for μ that are larger than the bifurcation values, the terms of the recursive sequence will begin to act differently. To explain the nature of bifurcation values, we will first examine the logistic map when μ is equivalent to these bifurcation values. When $\mu = 1$, we obtain a new equilibrium point of $\frac{\mu-1}{\mu}$

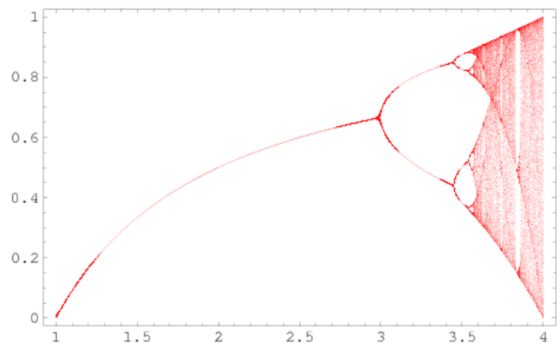

Figure 9: Diagram for the bifurcation values of the logistic map.

in addition to our zero equilibrium. This new equilibrium value is equal to our zero equilibrium because $\overline{x} = \frac{\mu-1}{\mu} = \frac{1-1}{1} = 0$, so we have repeated equilibrium points when $\mu = 1$. Once $\mu > 1$, we will generate a positive equilibrium value that is different from the zero equilibrium and has the general form $\frac{\mu-1}{\mu}$. If we look at the bifurcation diagram, the value of $\overline{x}$ is 0 when $\mu = 1$. Once μ is larger than 1, the new value of $\overline{x}$ changes to a positive quantity. Furthermore, the bifurcation diagram has one branch when $1 < \mu < 3$, which signifies that only the positive equilibrium value is locally asymptotically stable when $1 < \mu < 3$.

If $\mu = 3$, we acquire new values, namely the minimal period-2 solutions. At $\mu = 3$, the minimal period-2 solutions are the same as our equilibrium points of $\overline{x} = 0$ and $\overline{x} = \frac{2}{3}$. Once $\mu > 3$, the minimal period-2 solutions will be distinct from the equilibrium points, and the terms of the recursion will move closer to those minimal period-2 solutions. To connect this idea to Figure 9, when $\mu > 3$, the diagram branches into two stems, which indicates that the minimal period-2 solutions are the only stable attractors as μ passes through 3.

When $\mu = 1 + \sqrt{6}$, the function of the logistic map will generate minimal-period-4 solutions that are identical to the period-2 solutions. If μ surpasses $1+\sqrt{6}$, we will find new values that constitute the minimal period-4 solutions, and the recursion will approach those minimal period-4 solutions. If we look at Figure 9, the model splits into four branches when $\mu > 1 + \sqrt{6}$, which reinforces the notion that only the minimal period-4 values are locally asymptotically stable when μ is greater than $1+\sqrt{6}$. If $\mu \approx 3.544$, the function will have minimal period-8 solutions that are equivalent to the period-4 solutions. Once μ exceeds this approximation, the recursion will move towards the minimal period-8 solutions.

This pattern continues as we pass through each bifurcation value for the logistic map. This phenomenon is referred to as the *period-doubling bifurcation route to chaos* [1]. In other words, each time μ exceeds a bifurcation value of the logistic map, we multiply the period by 2, and the terms of the recursion will converge to those minimal periodic solutions. Mathematically, when μ passes through the k-th bifurcation value of the logistic map, the recursion will move towards the minimal period-$2k$ solutions.

One unique property that arises with the logistic map is *Feigenbaum's Constant*. The calculations needed for Feigenbaum's Constant can be visualized in Figure 10 [3]. The second column of the table shows a list of bifurcation values that are denoted as μ_n. The third column of the table subtracts the previous bifurcation value from the current bifurcation value, and the fourth column divides that quantity by the difference between the next and current bifurcation values. If we continue this procedure for each bifurcation value, then we will approach a value of approximately 4.6692. Algebraically, if we create a set $\{\mu_n\}$ of all the bifurcation values of the logistic map, then Feigenbaum's Constant is defined by: $\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1}-\mu_n} \approx 4.6692$ [3].

When we assign different values to the constant μ, the recursion of the logistic map converges to an equilibrium value or a set of minimal periodic solutions [1]. This trend continues until $\mu \approx 3.56994$. Once μ exceeds this value, the logistic map will experience chaotic behavior. *Chaos* refers to an expression where the terms of the recursion move in an unsystematic pattern, regardless of where we set the initial condition. Based on Figure 11, chaotic dynamical systems exhibit extreme sensitivity to initial conditions.

n	μ_n	$\mu_n - \mu_{n-1}$	$\frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$
0	3	—	—
1	3.449499 ...	0.449499 ...	—
2	3.544090 ...	0.094591 ...	4.752027 ...
3	3.564407 ...	0.020313 ...	4.656673 ...
4	3.568759 ...	0.004352 ...	4.667509 ...
5	3.569692 ...	0.00093219 ...	4.668576 ...
6	3.569891 ...	0.00019964 ...	4.669354 ...

Figure 10: Table of bifurcation values for the logistic map [3].

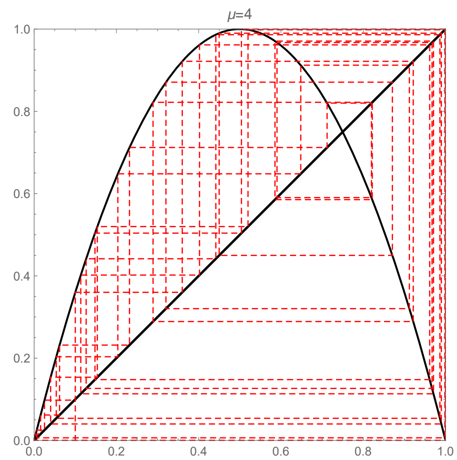

Figure 11: Graph of logistic map that exhibits chaos when $\mu > 3.56994$. In this case, we have $x_{n+1} = 4x_n(1 - x_n)$, where $\mu = 4$ and $x_0 = 0.1$. This image was developed by Dr. Elliott Bertrand with the use of Mathematica.

When we examine the logistic map, we work with unique solutions that originate from specified initial condition(s) and iterations of the difference equation. However, there are special cases where we can establish an *explicit solution* that provides a more convenient and efficient method for determining the n-th term of a recursive sequence. To show that an explicit solution is satisfied for a difference equation, we need to show that the next term of the recursion can be expressed as a function of the previous term of the recursion, $x_{n+1} = f(x_n)$, and that the explicit solution is satisfied for our initial condition x_0. For the logistic map, there are two values of μ that provide us with an explicit solution, $\mu = 1$ and $\mu = 4$ [4]. For $\mu = 4$, the logistic map will be $x_{n+1} = 4x_n(1 - x_n)$ and the explicit solution is $x_n = \frac{1}{2}(1 - \cos(2^n \cos^{-1}(1 - 2x_0)))$ [4].

We will first show that the explicit solution satisfies the recursion such that $x_{n+1} = f(x_n)$. Recall that when $\mu = 4$ the equation of the explicit solution is $x_n = \frac{1}{2}(1 - \cos(2^n \cos^{-1}(1 - 2x_0)))$ [4]. We can increase the index to see that $x_{n+1} = \frac{1}{2}(1 - \cos(2^{n+1} \cos^{-1}(1 - 2x_0)))$. To make these formulas more concise, we will let $p = \cos^{-1}(1 - 2x_0)$. Then we have $x_n = \frac{1}{2}(1 - \cos(2^n p))$, and $x_{n+1} = \frac{1}{2}(1 - \cos(2^{n+1} p))$. Plugging our formulas for x_n and x_{n+1} into the equation for the logistic map, we have

$$\frac{1}{2}\left(1 - \cos(2^{n+1}p)\right) = 4\left(\frac{1}{2}\left(1 - \cos(2^n p)\right)\right)\left(1 - \frac{1}{2}\left(1 - \cos(2^n p)\right)\right)$$

Simplifying the equation, we have $\frac{1}{2}\left(1 - \cos(2(2^n p))\right) = 2(1 - \cos(2^n p))\left(1 - \frac{1}{2}\left(1 - \cos(2^n p)\right)\right)$

$\iff \frac{1}{2}\left(1 - \cos(2(2^n p))\right) = 2\left((1 - \cos(2^n p))\right)\left(1 - \frac{1}{2} + \frac{1}{2}\cos(2^n p)\right)$

$\iff \frac{1}{2}\left(1 - \cos(2(2^n p))\right) = 2\left((1 - \cos(2^n p))\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2^n p)\right)$

$\iff \frac{1}{2}\left(1 - \cos(2(2^n p))\right) = \left(1 - \cos(2^n p)\right)\left(1 + \cos(2^n p)\right)$

$\iff \frac{1}{2}\left(1 - \cos(2(2^n p))\right) = 1 - \cos^2(2^n p)$

Recall that $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) = 1 - \cos^2(x)$. If we let $x = 2^n p$, then we can apply these trigonometric identities to obtain

$\frac{1}{2}\left(1 - \cos(2x)\right) = 1 - \cos^2(x) \iff \sin^2(x) = \sin^2(x)$

Next, we need to check that our explicit solution x_n is satisfied for any initial condition x_0, which will show us that the explicit solution is satisfied for any unique solution of the logistic map. If we plug $n = 0$ into our explicit solution, we find that $x_0 = \frac{1}{2}(1 - \cos(2^0 \cos^{-1}(1 - 2x_0))) = \frac{1}{2}(1 - \cos(\cos^{-1}(1 - 2x_0))) =$

$\frac{1}{2}(1-(1-2x_0)) = \frac{1}{2}(2x_0) = x_0$.

To provide an example of how this technique works, suppose we want to find the third term of a recursion where $\mu = 4$ and $x_0 = 0.2$. Since $\mu = 4$, we can use iterations of the logistic map, or we can utilize our explicit solution. We will first showcase the iterations that will return the third term of the sequence. Mathematically, we can write this operation as $f^3(0.2) = f(f(f(0.2)))$, which states that we are composing the logistic map with itself three times. Observe that

$x_1 = 4(.2)(1-.2) = .64$

$x_2 = 4(.64)(1-.64) = .9216$

$x_3 = 4(.9216)(1-.9216) \approx .289$

Using our explicit solution, we will set $n = 3$, since we are looking for the third term in the sequence.

Then $x_3 = \frac{1}{2}(1-\cos(2^3\cos^{-1}(1-2(0.2))))$

Simplifying the right side of the equation, we see that

$x_3 = \frac{1}{2}(1-\cos(8\cos^{-1}(0.6))) \approx \frac{1}{2}(1-\cos(425.041)) \approx \frac{1}{2}(0.578) = 0.289$

By exhibiting both techniques for evaluating the n-th term of a sequence, we see that the two approaches return the same value. The derivation of the case when $\mu = 1$ is similar. In this scenario, the explicit solution is $x_n = \frac{1}{2}\left(1-e^{(2^n\ln(1-2x_0))}\right)$.

In our study of difference equations, we can determine how parameters and variables influence the long-term behavior of recursive sequences. We can utilize equilibrium points and periodic solutions to understand when a difference equation follows a certain pattern. We can adjust the parameter(s) of a difference equation to see how the terms of the recursion shift to the equilibrium points or periodic solutions. For some difference equations such as the logistic map, we can expand into chaos theory by making minor adjustments to the constant(s). The terms of the recursive sequence will then experience an erratic and inconsistent trend, which allows us to draw unique implications from the parameter(s) within the difference equation.

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