Ancient Arabic Mathematics

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Abstract

Mathematics throughout history has been a continuous cycle of ideas building upon each other, none is more true of this idea than the ancient Arabic's dating back to the 9^{th} century. The ideology of the scientists and culture of that time brought together great minds that were able to uncover new ideas of how to solve quadratics and cubic equations. These ideas continued to inspire mathematicians centuries later.

1 The World in the 9^{th} Century

Towards the end of the 5^{th} century the Roman Empire was weakening its hold on the world and Western Europe was beginning its descent into the dark ages. Much of the world lost its ability to uncover the sciences, with the rise of Christianity the separation between culture and science grew. However, in one corner of the world, religion and science grew together and new advancements were being made everyday.

1.1 Ancient Arabic Culture

While Europe began the time of the dark ages, the Arabians began a period of expansion and enlightenment. This came with the rise of Islam, but before Islam rose, there were mainly nomadic Arab tribes inhabiting the land. As they began to unite they created the rich cultural life that we know today as Arabia.

While the Arabic land began to develop into society, the city of Baghdad was beginning to emerge. As society developed so did the religion of Islam and with the rise of this religion the need for science and mathematics in the Arabic world rose with it. Contrast to the spread of Christianity, Islam placed a heavy emphasis on science. Not only was a it a requirement of Muslims to search for enlightenment but the religion required mathematicians and astronomers to determine directions, track the moon phases and determine times of prayer [2].

A huge surge in the culture of scholarship began in the mid-eighth century within the Islamic Empire was started with the Translation Movement. This movement began in Baghdad, the cultural hub of the Arab's, and was the beginning of translating works from the earlier Greeks, Persians, and Indians into Arabic. This movement was helped by the patronage of the elite within the society. This movement created an intellectual atmosphere within Arabic society, primarily in Baghdad, that would inspire further scholarship and developments within their society for centuries to come.

1.2 The House of Wisdom - "Bayt al-Hikma"

In a response to the Translation Movement and an inspired academic himself, one caliph (ruler) of the Arabic Empire, al-Ma'mun wanted his city to become an intellectual hub.

The bastard son of Harun al-Rashid, the caliph of the Abbasid Empire in the mid-8th century, born year 786 AD the son of a caliph and a kitchen slave. Al-Ma'mun ended up ruling the Abbasid Empire after the death of his father in the early 9th century. In his younger years he was instilled with a love of scholarship, a need to learn.

Once he became a caliph himself, he played a major factor in the creation of the House of Wisdom. Al-Ma'mun was taken up by the Translation Movement and decided that Baghdad needed to be an intellectual hub. He sent emissaries out to gather the ancient texts of the Greeks and Indians. Another effort he made was after he defeated foreign rulers in battle, rather than surrendering their gold and wealth, he required them to surrender their libraries [2].

1.3 Arabic Mathematics

The basics of math within the time of the ancient Arabic's had some interesting quirks. To start with, arithmetic had not yet been developed in their culture so all math was done entirely in words. For example the equation:

$$8x^2 = 3x + 2$$

would be said: eight properties are equal to three things plus two dirhams, if in reference to money.

Some other mathematical beliefs back in the time of the ancient Arabic's include their ideas on negative numbers. Plainly, they did not believe in negative numbers. To the ancient Arabic's only positive numbers made sense to them. Numbers began as measures of counting, to count livestock and items for trade or sale. Numbers represented physical items and you could not have a negative of a physical item [4]. When it came to more complicated math and the studies of mathematicians within the House of Wisdom, they had the studies of the Greeks and Indians before them, both which gave contradictory claims on the validity of negative numbers. Because of this the Arabic mathematicians were split on this idea, some split these ideas down the middle, like al-Khwarizmi, who decided to look at negative numbers as a debt rather than a negative quantity.

2 Arabic Mathematicians

The ancient Arabic mathematicians during the time of the House of Wisdom were jacks of all trades. They could be considered mathematicians, astrologers, translators, philosophers, poets, and engineers all wrapped in one.

2.1 Al-Khwarizmi

Born Muhammad ibn Musa al-Khwarizmi in 780 AD and died 850 AD there is little known of his actual life, only the mathematical legacy he left behind. Back in ancient Baghdad, al-Khwarizmi was employed by al-Ma'mun in his House of Wisdom as a mathematician and astronomer, it is important to note that he was not employed as a translator of ancient texts by al-Ma'mun but rather as a researcher.

Al-Khwarizmi wrote the book on algebra in his time. The term "algebra" is supposedly derived from the title, his book *al-Kitab al-kukhtasar fihisab al-jabr* *wa'l-muqabala* or to put it simply *Kitab al-Jabr* meaning "The Compendious Book on Calculation by Completion and Balancing" which set the guidelines for algebra in Islamic mathematics.

The *Kitab al-Jabr* lays out for the first time the rules and steps of solving algebraic equations. It is layed out in three parts; the first devoted to solving equations, the next to practical measuring such as areas and volumes, and the last devoted to problems created by the complicated Islamic laws of inheritance involving arithmetic and linear equations. In traditional Arabic mathematical fashion, everything in this book is expressed in written words, no arithmetic used. In this book al-Khwarizmi did not acknowledge the existence of negative numbers or the possibility of zeros as coefficients. He gave proofs showing his methods worked, something that was not a standard in his time, and more interestingly these proofs were geometrical [3].

Al-Khwarizmi opens his book by discussing equations of the first and second degree. He studied these equations not just as ways for solving equations but studied them in their own right by classifying them into six different types of equations:

$$ax^2 = bx, \quad ax^2 = b, \quad ax = b, \quad ax^2 + bx = c, \quad ax^2 + c = bx, \quad ax^2 = bx + c$$

where a, b,and c are positive integers. [1] These different types of equations were needed because he did not recognize the existence of negative numbers.

Al-Khwarizmi's work in quadratics helped evolve the subject into what we know of it today. What we now know today as the completing the square method was originally developed by al-Khwarizmi geometrically. He used squares to show his method.

Today we solve quadratic equations through the completing the square method and the quadratic equation. For reference the quadratic formula states that:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

and we can prove this.

Proof: For any quadratic equation like $ax^2 + bx + c = 0$. First start by completing the square.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \square$$

For example, take the equation:

$$x^2 + 10x = 39$$

Today we would solve that through the completing the square method through the steps of:

$$x^2 + 10x + 5^2 = 39 + 5^2$$

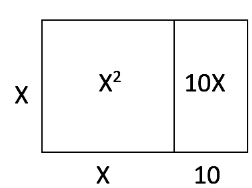
$$x^2 + 10x + 25 = 64$$

$$\sqrt{(x+5)^2} = \sqrt{64}$$

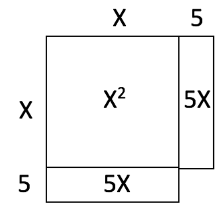
$$x + 5 = \pm 8$$

$$x = 3, -13$$

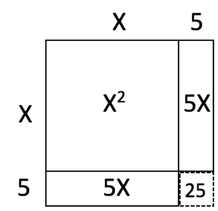
This is a method we are all familiar with, and this comes from al-Khwarizmi. Al-Khwarizmi would solve this equation geometrically through these steps: First draw a squares with sides x, so the area of the of the square is x^2. Then draw a rectangle alongside the square with one side equaling x and the other side equaling 10, the area of this rectangle would be $10x$. The equation tells us that the area of the whole figure is 39.



Next we split the rectangle into two halves with an area of $5x$. Then move one of these rectangles to the bottom of the figure.



This leaves a small square missing in the corner of the figure. Since the sides of the rectangles are equal to 5, the area of the small square is equal to 25.



This would bring the whole figures area to equal 64, since $25 + 39 = 64$. This makes the whole area a square. This means the each side of the figure is equal to the square root of 64, which is 8. Since the side of the figure is $x + 5$ we can conclude that $x + 5 = 8$, When solving for x we get $x = 3$ and thus the equation $x^2 + 10x = 39$ is solved. By doing this method we are in fact completing the square.

2.2 Al-Khayammi

Omar Khayyam (1048-1131), also known as al-Khayammi, was a poet, philosopher, astronomer, and mathematician. He devoted his mathematical work to solving quadratic and cubic equations through geometrical constructions. His solutions were obtained by coordinates of the points where curves are met, for example a circle and a parabola. However, one limitation in his work is that there was no possibility in getting any numerical solutions through his geometrical constructions.

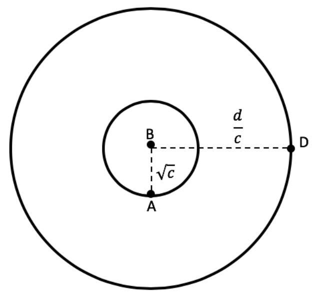
Al-Khayammi wrote many books in his time, the goal of many of his book's was to be able to solve equations of x^3. He was able to obtain solutions by using parabolas and hyperbolas to determine points. But finding a geometric solution wasn't helpful if one wanted to find a number that solves the equation [3].

Al-Khayyam solved cubic equations geometrically. First he would take the equation

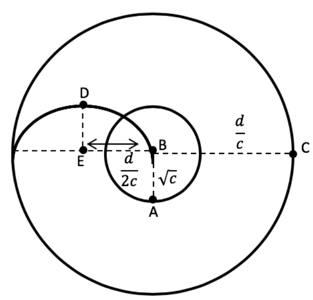
$$x^3 + cx = d$$

where c, d are given positive constants. Then construct a circle with radius of $\sqrt{c}$. Then make another circle with a radius of $\frac{d}{c}$, this circle can be bigger or smaller than the initial circle, in this case we will make it bigger.

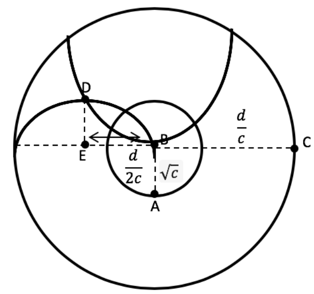
Next we pick a point within the first circle, lets call it B. Lets draw a dotted line from point B to a point A along the edge of the circle, this will be the length $\sqrt{c}$.



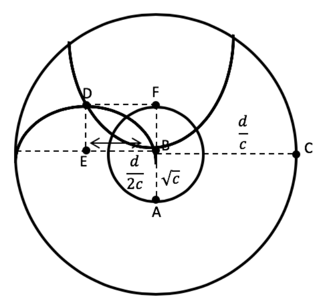
Next, construct a half circle from point B to the out edge of the large circle. The radius of this half circle being $\frac{d}{2c}$.



Then, we construct a parabola using the equation $y = \frac{x^2}{\sqrt{c}}$. It will intersect the half circle at some point D. Then draw a straight line from this point D to another point E, it does not have to be point E which is halfway across the half circle but in this case it will be.



We can then connect the points E to B and respectfully D to F. The length of the line EB is the solution for x since it is perpendicular to the line DE, which intersects both the half circle and parabola $y = \frac{x^2}{\sqrt{c}}$.



The problem with al-Khayammi's solution is that it can only find one solution to the equation $x^3 + cx = d$ when the function really can have up to three real solutions and the solution is the length between two points and not a numerical solution.

Al-Khayammi's solutions left much to be desired and this was the goal of many mathematicians for centuries to come.

3 Implications for Modern Algebra

3.1 The 16^{th} Century

Al-Khayammi's work with cubic equations left a lot to be solved. This is an example of how the work of the ancient Arabic mathematicians bled into later areas of mathematics. In the 16^{th} century Tartaglia and Scipione del Ferro attempted to solve the problem al-Khayammi had left behind with cubic equations. During this time a lot of math was kept secret and used by mathematicians in "duels" where two mathematicians would attempt to out-smart each other through their mathematical abilities. This caused a lot of strife and secret keeping in the mathematical community, Tartaglia and del Ferro were two mathematicians put against each other in an attempt to solve the cubic equation. They both claimed to have solutions to this problem:

$$ax^3 + bx^2 + cx + d = 0$$

and both nearly took the answers to their graves. The person who brought to light these answers, specifically Tartaglia's, was Girolamo Cardano. Tartaglia had told Cardano his solution to the cubic equation in the form of a poem in the mid 1500's, translated from Italian to English. A few lines can be picked out from this poem in order to make sense of and solve the cubic equation:

"When the cube with the things together: $x^3 + px$

Are equal to some discrete number: $= q$

Find two other numbers differing in this one: $u - v = q$

That their product will should always be equal: $uv =$

Exactly to the cube of a third of the thing: $(p/3)^3$

Of their cube roots subtracted: $\sqrt[3]{u} - \sqrt[3]{v}$

Will be equal to your principal thing: $= x$

When the cube remains alone: $x^3 = px + q$

You will at once divide the number into two parts: $q = u + v$

So that the one times the other produces clearly: $uv =$

The cube of a third of the things exactly: $p^3/3$ instead of $(p/3)^3$

You will take the cube roots added together: $\sqrt[3]{u} + \sqrt[3]{v}$

And this sum will be your thought: $= x$

The third of these calculations of ours: $x^3 + q = px$

Is solved with the second if you take good care."

Cardano promised Tartaglia that he would not reveal to anyone his solution to the cubic equation, but Cardano betrayed this and ended up publishing the findings, alongside his own conclusions from other research, in 1545 in his book *Ars Magna* and thus betrayed his oath with Tartaglia through a loophole he found.

While it may seem that Cardano solved the problem of cubic equations, the solution to such has been named "Cardano's formula", he did not discover it. He and his disciple, Lodovico Ferrari, examined the personal notes of Scipione del Ferro and discovered that he indeed had the formula all along [3].

Scipione del Ferro's formula being:

THEOREM 3.1 *Equation:* $x^3 + px = q$

Discriminant: $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$

Root: $x = \sqrt[3]{\sqrt{\Delta} + \frac{q}{2}} - \sqrt[3]{\sqrt{\Delta} - \frac{q}{2}}$

This can be proved by:

Proof: Compute the cube of a sum:

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

Assume x has an expression as a sum of two cube roots;

$$x = \sqrt[3]{u} + \sqrt[3]{v}$$

Then if we take

$$a = \sqrt[3]{u}, \qquad b = \sqrt[3]{v}$$

and plug this back into our original formula, we get:

$$x^3 = u + v + 3\sqrt[3]{u}\sqrt[3]{v}(\sqrt[3]{u} + \sqrt[3]{v})$$

$$= [u+v] + 3\sqrt[3]{u}\sqrt[3]{v}x$$

So

$$x^3 + [3\sqrt[3]{u}\sqrt[3]{v}]x = u + v$$

Then we must take

$$u + v = q, \qquad -3\sqrt[3]{u}\sqrt[3]{v} = p$$

By cubing the second formula above we then get:

$$u + v = q, \qquad uv = -\tfrac{p^3}{27}$$

Therefore we can say that u and v are the roots of the quadratic equation in y,

$$y^2 - qy - \tfrac{p^3}{27} = 0$$

This is solved, giving the formula for u and v:

$$u, v = \tfrac{q}{2} \pm \sqrt{\Delta}, \qquad \Delta = \tfrac{q^2}{4} + \tfrac{p^3}{27}$$

Since

$$x = \sqrt[3]{u} + \sqrt[3]{v}$$

Then, when substituted, the expressions for u, v give:

$$x = \sqrt[3]{\sqrt{\Delta} + \tfrac{q}{2}} - \sqrt[3]{\sqrt{\Delta} - \tfrac{q}{2}} \qquad \square$$

Thus the cube root is solved! And finally, numerical solutions to the problem can be found. Thus finishing up and closing the door to al-Khayammi's finished but unsatisfactory approach to finding equations of degree 3.

Later, Lodovico Ferrari, Cardano's disciple attempted to solve biquadratic equations. These come in the form:

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

Ferrari found a way to solve this using a depressed quartic equation. This is the equation of the form:

$$x^4 + ax^2 + bx + c = 0$$

Ferrari solved this by rewriting the depressed quartic by expanding the square and regrouping all the terms to the left-hand side. This get's us:

$$(x^2 + \frac{a}{2})^2 = -bx - c + \frac{a^2}{4}$$

Ferrari introduces a variable u on the left-hand side by adding $2x^2u + au + u^2$ to both sides. This gives the equation:

$$(x^2 + \frac{a}{2} + u)^2 = 2ux^2 - bx + u^2 + ua + \frac{a^2}{4} - c$$

This is equivalent to the original equation when a value is given to u. Since this value can be arbitrarily chosen, Ferrari chose it to complete the square in the right hand side. Implying that the discriminant in x is zero and u is the root of the equation:

$$(-b)^2 = 4(2u)(u^2 + au + \frac{a^2}{4} - c) = 0$$

This can be rewritten as:

$$8u^3 + 8au^2 + (2a^2 - 8c)u - b^2 = 0$$

This is known as the *auxiliary cubic equation*. Ferrari let u be the root of this equation. Therefore the equation becomes:

$$(x^2 + \frac{a}{2} + u)^2 = (x\sqrt{2u} - \frac{b}{2\sqrt{2u}})^2$$

Each of these equations when solved will lead to two roots, so there is a method of finding the four roots of the original biquadratic equation [3].

3.2 The 19^{th} Century

The desire to discover a way to solve equations of degree greater than four continued. After Ferrari's discovery of how to solve for bi quadratic equations it became the question for mathematicians on how to solve for equations of degree 5 or higher, but this question proved harder to solve than anticipated. Until in the 19'th century this was proved impossible by mathematicians; Abel, Galois, and Ruffini. Paolo Ruffini and Niels Abel formulized a theorem to prove that no equation of degree 5 or higher could be solved.

THEOREM 3.2 *Let $n \geq 5$. Then there exist $a_0, \ldots, a_{n-1}$ in $\mathbb{C}$ such that no root in $\mathbb{C}$ of the equation $x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ can be obtained from $\{0, 1, a_0, \ldots, a_{n-1}\}$, in a finite number of steps, using the operations $+, -, \cdot and/, and()^{1/r}$ (with choice) for all $r \geq 1$ in the integers.*

Ruffini proved this first back in 1799, but the permutations in his proof were not understandable and not widely accepted by the mathematical community. Abel later attempted to prove it in 1823 but again it was very hard to understand and not widely accepted by the mathematical community, later the proof was clarified by Galois in 1830 using Galois Theory.

While many have tried to create proofs for this theorem, the most widely accepted proof for this theorem is based on Galois theory, which is concerned with permutations (one-to-one mapping of a set onto itself) in the roots of polynomials. This proof is stating that an equation is solvable in radicals if and only if it has a solvable Galois group, which is a specific group associated with a certain type of field extension. Therefore the proof of the Abel-Ruffini theorem comes down to computing the Galois group of the general polynomial of the fifth degree [5]. The concept of Galois theory and the use of Galois theory in the proof of this theorem extends beyond the mathematics of this paper. But to give a general idea of the Galois proof of this theory:

One must compute the Galois group of the general equation of the n^{th} degree and show it is equal to S_n, which is the group of permutations on n objects. Galois theory asserts that the polynomial equation is solvable if and only if its group is solvable. Thus you must show that S_n is not a solvable group when $n \geq 5$.

References

[1] T. Gowers, J. Barrow-Green, I. Leader. The Princeton Companion to Mathematics. *The Princeton Companion to Mathematics*, 2010.

[2] Jim al-Khalili. The House of Wisdom: How Arabic Science saved Ancient Knowledge and Gave us the Renaissance. *Penguin Book*, 2010.

[3] V.S. Baradarajan. Algebra in Ancient and Modern Times. *American Mathematical Society and Hindustan Book Agency*, volume 12, 1998.

[4] William Berlinghoff, Fernando Gouvea. Math through the Ages: A gentle history for teachers and others. *Oxten House Publishers*, 2002.

[5] Emil Artin. Galois Thoery. *Edwards Brothers, Inc.*, Notre Dame Mathematical Lectures, number 2, 1942.