In the race towards infinity who wins: Polynomials or exponentials?

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Abstract

Part of understanding the global dynamics of mathematical models is to investigate the end behaviors (i.e. limits at infinity) of these models. As shown in almost every precalculus course, values of both exponential functions of the form e^{kx} , k > 0, and polynomials with positive leading coefficient grow as their input values gets “arbitrarily large”. Motivated by these facts, we investigate how the growth of exponential functions compares to the growth of polynomials. In particular, we show that every function of the form e^{kx}, for k > 0, eventually dominates every polynomial.

Objective

We shall show that every exponential function of the form e^{kx}, for k > 0, eventually dominates every polynomial; that is , for all polynomials $P_n(x)$ of degree n, we have

$$\lim_{x\to\infty} \frac{P_n(x)}{e^{kx}} = 0, \text{ for } k > 0.$$

Definitions, Theorems and Known Proofs

1. Polynomials: a polynomial of degree n is a function of the form :

$a_nx^n+a_{n-1}x^{n-1}+\ldots\ldots\ldots+a_1x+a_0$, where $a_n \neq 0$.

Dominant term analysis tells us that lim $P_n(x)$ behaves like lim a_nx^n as x approaches $\pm\infty$. Hence, for polynomials with positive leading coefficient a_n, we have $P_n(x) = a_n$ for n = 0 and lim $P_n = +\infty$, for n > 1, as x approaches $+\infty$.

2. Eventually dominates: an exponential growth function goes to infinity at a faster rate compared to a polynomial.

3. End behaviour: the behaviour of a function as x approaches plus or minus infinity. For example, for k > 0, $\lim e^{kx} = +\infty, \text{as } x \to +\infty$

4. L'Hospital Rule: at a given point if two functions have an infinite limit or zero as a limit and are both differentiable in a neighbourhood of this point then the limit of the quotient of the functions is equal to the limit of the quotient of their derivatives provided that this limit exists.

5. Power rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

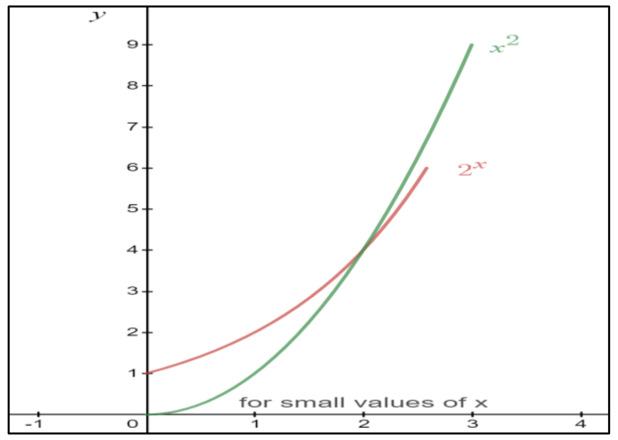
It follows by induction and linearity property of differentiation that all polynomials are differentiable everywhere [3].

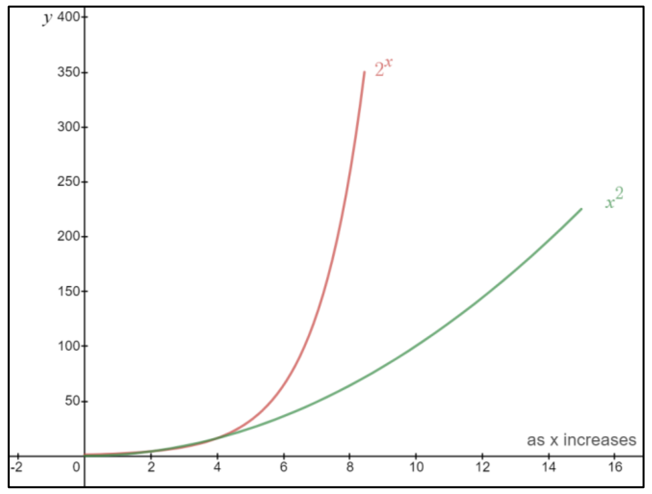
6. Derivative of exponential functions:

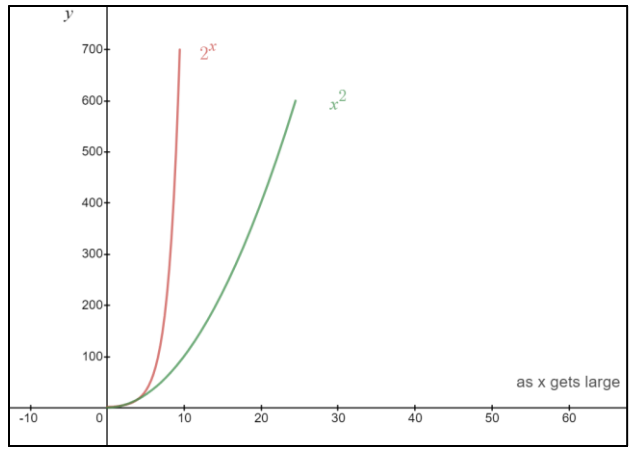
$$\frac{d}{dx}(e^{kx}) = ke^{kx}$$

It follows that all exponentials are differentiable everywhere [3].

Graphical representations







Proof

We prove that:

$$\lim_{x\to\infty} \frac{P_n(x)}{e^{kx}} = 0, \text{ for } k > 0$$

We proceed with Induction,

- Case 1: n = 0 implies $P_n(x)=C$, where C is a constant.

  Thus, $\lim_{x\to\infty} \frac{C}{e^{kx}} = 0$

- Case 2: n = 1 implies $P_n(x)=mx+b$, $m\neq 0$.

  Thus, $\lim_{x\to\infty} \frac{P_n(x)}{e^{kx}} = \lim_{x\to\infty} \frac{mx+b}{e^{kx}}$
  $= \lim_{x\to\infty} \frac{m}{ke^{kx}} = 0$ by 4.

<u>Induction Hypothesis</u>: assume that

$$\lim_{x\to\infty} \frac{P_n(x)}{e^{kx}} = 0, \text{ for all } n;$$

We shall show that it is also true for *n+1.*

From 4,5,6:

$$\lim_{x\to\infty} \frac{P_{n+1}(x)}{e^{kx}} = \lim_{x\to\infty} \frac{P_n(x)}{ke^{kx}}$$
$$= \frac{1}{k} \lim_{x\to\infty} \frac{P_n(x)}{e^{kx}}$$
$$= \frac{1}{k} * 0 = 0 \text{ by induction hypothesis.}$$

Hence, our desired result. Q.E.D.

Conclusion

Our result can be extended to all exponential growth functions by combining 3, 4, and 6.

References, acknowledgements

- [1]. Single variable calculus, by James Stewart, 2015.
- [2]. Calculus 1 with Precalculus: A One-year Course, by Ron Larson, 2002.
- [3]. An introduction to Real Analysis, by John K.Hunter, 2012

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