History of Algebra and its Implications for Teaching

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Abstract

Algebra can be described as a branch of mathematics concerned with finding the values of unknown quantities (letters and other general symbols) defined by the equations that they satisfy. Algebraic problems have survived in mathematical writings of the Egyptians and Babylonians. The ancient Greeks also contributed to the development of algebraic concepts. In this paper, we will discuss historically famous mathematicians from all over the world along with their key mathematical contributions. Mathematical proofs of ancient and modern discoveries will be presented. We will then consider the impacts of incorporating history into the teaching of mathematics courses as an educational technique.

1 Introduction

In order to understand the way algebra is the way it is today, it is important to understand how it came about starting with its ancient origins. In a modern sense, algebra can be described as a branch of mathematics concerned with finding the values of unknown quantities defined by the equations that they satisfy. Algebraic problems have survived in mathematical writings of the Egyptians and Babylonians. The ancient Greeks also contributed to the development of algebraic concepts, but these concepts had a heavier focus on geometry [1]. The combination of all of the discoveries of these great mathematicians shaped the way algebra is taught today. It can be important to consider how history is or can be incorporated into the teaching of it in school systems. In this paper, we will start off by discussing famous mathematicians from around the world and their contributions in section 2. In section 3 proofs of theorems discovered by some of these famous mathematicians, specifically from ancient times, will be presented. In section 4 more proofs will be presented, but these proofs will involve concepts from more modern times. Lastly, in section 5 we will discuss the implications for teaching.

2 History and Background

2.1 Greek Mathematicians

One highly influential ancient Greek mathematician was Pythagoras. Pythagoras lived during the sixth century B.C and was a great mathematician as well as a philosopher. Over time he gained many followers known as Pythagoreans. He has been given credit for and is most well known for his theorem known as the Pythagorean theorem [2]. This theorem states that the sum of the squares of the lengths of the two shorter sides of a right triangle is equal to the square of the length of the longer side (hypotenuse) of the triangle. In its most common form, the theorem states that $a^2 + b^2 = c^2$ where a and b are the lengths of the legs of the right triangle and c is the length of the hypotenuse [3]. Not only is Pythagoras known for this work with right triangles, but the Pythagoreans are also known for the discovery of irrational numbers. Geometrically speaking, if AB and CD are two segments, they are considered commensurable if there is a unit EF such that both AB and CD are multiples of EF, so that $AB : CD = m : n$ where m and n are positive integers. What the Pythagoreans discovered was that the side and the diagonal of a square are not commensurable. If the side of a square has length 1, this implies that the diagonal, which has length $\sqrt{2}$, is not commensurable with this side. In other words, $\sqrt{2}$ is not a rational number [2]. This proof will be shown in section 3 of the paper. Pythagoras certainly expressed how the world is highly mathematical and initiated this new way of thinking [3]. He was able to make contributions that would aid in future algebraic discoveries.

Another influential Greek Mathematician was Euclid. Euclid is one of the most famous mathematicians of all time. He lived in Alexandria Greece around 300 B.C. He was able to establish Geometry as a deductive science based on a small number of fundamental principles called axioms [2]. His major work where he presented these ideas is called *Elements*. Euclid's *Elements* is said to have been one of the most influential mathematical treatises. Although his work does not contain much original content, the order of propositions and overall logical structure of his work is what made it stand out. Some of the most famous of the theorems in the Elements are the following:

The sum of the three angles of a triangle is equal to two right angles.

The area of the square on the hypotenuse of a right angled triangle is equal to the sum of the areas of the squares on the other two sides.

Euclid's *Elements* contained the "windmill proof" of the Pythagorean theorem. This proof used definitions regarding things such as lines, right angles, parallel lines and self evident postulates. He also included a proof of the converse of this theorem.

2.2 Persian Mathematicians

Al-Khwarizmi was a key Persian mathematical figure who introduces the concepts of algebra into European mathematics. He was one of the most influential figures of his time and was a central figure in the separation of Algebra from Geometry and its development as an independent discipline. He was able to classify quadratic equations and even gave methods to find solutions for each type [2]. In 825 CE, Al-Khwarizmi discovered a general method that is similar to today's Quadratic Formula. He gave a method to solve any equation of the form $aX^2 + bX = c$, where a,b and c are positive numbers [8]. He did not consider equations of the form $aX^2 + bX + C = 0$ because he considered only positive numbers. If a,b, and c were all positive numbers then this equation would have no positive solutions. An Italian mathematician who we will talk about later on in the paper, Gerolamo Cardano, combined Al-Khwarizmi's solution with geometry to solve quadratic equations. Cardano allowed for negative solutions in his method. There was also a French mathematician, Rene Descartes, who actually published the Quadratic Formula as it exists today in his work.

2.3 Italian Mathematicians

After discovering solutions to quadratic equations, mathematicians around the world were struggling to come up with a solution to third power equations. Scipione Del Ferro was an Italian mathematician who made great achievements in this area. He lived from 1465 to 1526 and was a professor of mathematics for thirty years at the University of Bologna [2]. Unfortunately, he did not publish his discoveries so he wasn't given too much credit. He was able to solve the equation:

$$X^3 + PX = Q$$

This form is known as the depressed cubic form where P and Q are coefficients. In this case the coefficient of the quadratic or X^2 term is 0. All cubic equations can be reduced to this form, but mathematicians during this time were only familiar with the positive solutions.

During these times, mathematicians assumed both P and Q to be positive numbers. The formula that Del Ferro obtained only works using this assumption. His method works with cubic equations only in this depressed form. Del Ferro's formula can be given in the following form:

$$\text{Equation: } X^3 + PX = Q$$

$$\text{Discriminant: } \Delta = \frac{Q^2}{4} + \frac{P^3}{27}$$

$$\text{Root: } X = \sqrt[3]{\sqrt{\Delta} + \frac{Q}{2}} - \sqrt[3]{\sqrt{\Delta} - \frac{Q}{2}}$$

The discriminant of any polynomial is a quantity that depends on the coefficients and determines various properties of the roots. For both quadratic and cubic equations, the roots are real and distinct if the discriminant is positive, they are real with at least two equal if the discriminant is equal to zero, and they include a conjugate pair of complex roots if the discriminant is negative. After Del Ferro discovered his solution, another Italian mathematician known as Gerolamo Cardano made significant contributions as well. Cardano lived from 1501 to 1576 and was a Renaissance scholar and physician as well as a mathematician [2]. He is famous as the author of $ArsMagna$. This is a book that marked the beginnings of the subject of modern algebra. In this book he describes Del Ferro's formula. Cardano was the first to actually publish the solution to the cubic equation. The difference is, that Cardano made the solution work for values of P amd Q which did not necessarily have to be positive. It is valid irrespective of the signs of P and Q. Cardano's formula is as follows:

$$\text{Root: } X = \sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}$$

Formally, it is the same as del Ferro's formula, except for the + sign in the second term which can be explained by reversing the sign of the expression in the second cube root in del Ferro's formula [2].

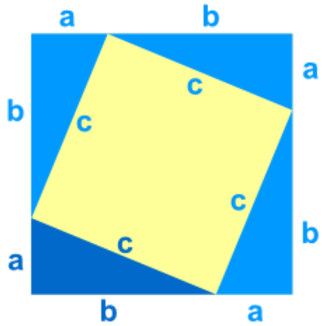
After the cubic solution had been presented, the next main focus would be on the biquadratic or quartic solution. Lodovico(Luigi) Ferrari is another Italian mathematician who was actually the first to find an algebraic solution for equations of this type in 1540. Ferrari worked in Cardano's household as a servant.

Cardano eventually realized how exceptional Ferrari was and quickly began to treat him as a disciple and later on as a collaborator. Ferrari was able to learn the solution of the cubic equation from Cardano and eventually made his own contributions when he discovered the method of solving the biquadratic, also known as the quartic, equation. This method involves reducing the original equation to the solution of an auxiliary cubic equation, which could then be solved using Cardano's method [2]. Ferrari's method requires removing the X^3 term by substituting an expression in for X. The value for X that works is $X = Y - \frac{A}{4}$. After making this substitution, Ferrari was able to get an expression of the form $X^4 + PX^2 + QX + R = 0$ where P,Q and R are new constants.

3 Ancient Algebra

Ideas discovered by mathematicians from ancient times are still taught in math courses today. One example is the Pythagorean theorem. Although a proof of the Pythagorean theorem wasn't believed to be provided by Pythagoras himself, a proof certainly exists today. The theorem specifically states:

Theorem 3.1. *In a right-angled triangle, the square of the hypotenuse side is equal to the sum of squares of the other two sides such that* $a^2 + b^2 = c^2$.



Proof. First, let the length of the sides of the larger square be equal to a+b and let the sides of the smaller square have length c. Now, let us consider the area of the larger square. The total area can be represented by:

$$A = (a + b)^2 = (a + b)(a + b).$$

Now, lets consider the areas of the smaller square and the triangles formed. The area of the square of side length c can be represented by c^2.

For each of the four triangles, the area of each can be represented by:

$$\frac{ab}{2}$$

Since there are 4 triangles, multiply by 4 to get

$$\frac{4ab}{2} = 2ab.$$

Now, we can add these two areas to get

$$A = c^2 + 2ab.$$

Since we know that the area of the large square and the sum of the area of the smaller square and triangles combined are equal, we can write

$$(a + b)(a + b) = c^2 + 2ab$$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

Finally we have,

$$a^2 + b^2 = c^2$$

□

The Pythagorean theorem is a critical concept for students to learn. This theorem is typically introduced during the middle school years, but becomes increasingly more important during the high school years. Presenting a formal proof to the students prior to showing examples allows them to gain insight as to where the formula they are using came from. It is essential that students understand the geometric concepts behind the theorem as well as its algebraic representation. Presenting a proof is one way of accomplishing this [4]. Doing so may aid in the student's understanding by allowing them to see the connections.

Euclid created another proof of the Pythagorean theorem. He created a geometry based proof of the theorem as well as a proof of the converse of the theorem.

This proof is contained in Euclid's famous work *Elements*. His proofs use definitions, postulates and propositions that he had previously shown to be true [5]. In ancient Greece, discoveries involving irrational numbers were made, which would eventually become a core part of learning mathematics. As mentioned above, Pythagoras began the thinking that led to the discovery of irrational numbers. The Pythagoreans considered the situation where the lengths of the sides of a square were 1. According to the Pythagorean theorem, this would mean the length of the diagonal would be $\sqrt{2}$. We will prove that the $\sqrt{2}$ is irrational, but first we must prove that if for some integer a we have that a^2 is even then a must also be even.

Lemma 3.2. *Suppose a is an integer. If a^2 is even then a is even.*

Proof. Using proof by contrapositive, assume that a is an odd integer. Then, using the definition of odd integers, we can say that $a = 2k + 1$ where k is another integer. Then $a^2 = (2k+1)(2k+1) = 4k^2+4k+1 = 2(2k^2+2k)+1$. Since $2k^2+2k$ involves the sum and product of integers, it is also an integer. This implies that a^2 is also odd. By proof by contrapositive, if a^2 is even, then a is even. □

This result will be used in the following proof.

Theorem 3.3. *$\sqrt{2}$ is an irrational number.*

Proof. We will use proof by contradiction. Assume that $\sqrt{2}$ is rational. Then we have that $\sqrt{2} = a/b$, where a and b are integers, $b \neq 0$ and without loss of generality, we can assume that $\frac{a}{b}$ is in lowest terms. We can square each side to get $2 = \frac{a^2}{b^2}$. Multiplying each side by b^2 we get $2b^2 = a^2$. From this, we know that a^2 must be an even number since it can be written as 2 times an integer. Then by lemma 3.2, a itself must be an even number. Since a is an even number, then we can write $a = 2m$ where m is an integer. By substituting $a = 2m$ into the equation $2 = \frac{a^2}{b^2}$ we get $2 = \frac{(2m)^2}{b^2} = \frac{4m^2}{b^2}$. Now multiply by b^2 to get $2b^2 = 4m^2$. Since m^2 is an integer, we know that b^2 must be even which also implies that b is even from lemma 3.2. Since 2 is a factor of both a and b, $\frac{a}{b}$ can not be in lowest terms. This is a contradiction since we said that a/b is in lowest terms. Thus, $\sqrt{2}$ is irrational.

□

4 The Rise of Modern Algebra

The history of solving quadratic equations stretches as far back as the Old Babylonian Period which was around 2000-1600 B.C. [7]. Many different mathematicians made contributions to this topic. A formula to solve quadratic equations that we use today, is known as the quadratic formula. This formula would later become a standard part of teaching a first course in Algebra.

Theorem 4.1. *Assuming* $a \neq 0$, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ when $ax^2 + bx + c = 0$.

Proof. Consider $ax^2 + bx + c = 0$ where a,b and c are real numbers. First, since

$a \neq 0$ we can divide the quadratic equation by a to get the following equation

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

Let $x = y + d$. We will choose d later on to make the resulting equation easier to solve. After substituting, we get

$$(y + d)^2 + \frac{b}{a}(y + d) + \frac{c}{a} = 0$$

After multiplying out we get

$$y^2 + 2yd + d^2\frac{b}{a}y + \frac{bd}{a} + \frac{c}{a} = 0$$

where $\frac{bd}{a}$ and $\frac{c}{a}$ are constants

We can also write this as

$$y^2 + (2d + \frac{b}{a})y + constants = 0$$

Notice that if the y coefficient were zero, we could move the constants to the other side of the equation and solve for y by taking the square root. We can then find y easily if we let $d = \frac{-b}{2a}$.

$$y^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$y^2 = \frac{b^2 - 4ac}{4a^2}$$

The quantity $b^2 - 4ac$ is known as the discriminant of the quadratic, D. We can write:

$$y = \pm\tfrac{\sqrt{D}}{2a}$$

Then we have that

$$x = d + y = \tfrac{-b}{2a} \pm \tfrac{\sqrt{D}}{2a} \text{ for } D > 0$$

This can then be written as

$$x = \tfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

□

For the above proof, note that if $D < 0$ we can write x in the complex form

$$x = d + y = \frac{-b}{2a} \pm \frac{\sqrt{-D}}{2a} i$$

After understanding quadratic equations, we can now look into cubic equations. After the major contributions by Scipione Del Ferro and Gerolamo Cardano, there now exists a formula and a proof that works for solving cubic equations. The method shown below uses the process of removing the X^2 term in order to simplify the equation. It took centuries of experience for mathematicians to come up with the tricks required to complete the following proof.

Theorem 4.2. $X = \sqrt[3]{\sqrt{\Delta} + \frac{Q}{2}} - \sqrt[3]{\sqrt{\Delta} - \frac{Q}{2}}$ *for equations of the form* $aX^3 + bX^2 + cX + d = 0$ *when* $a \neq 0$.

Proof. Consider $aX^3 + bX^2 + cX + d = 0$ where a, b, c, and d are real numbers such that a$\neq$ 0.

First we can divide by a to obtain the following cubic equation

$$X^3 + \frac{b}{a}X^2 + \frac{c}{a}X + \frac{d}{a} = 0$$

Next, we can substitute $X = y + e$. Let $e = \frac{-b}{3a}$ to get $X = y - \frac{b}{3a}$. By doing this and working out the algebra, we obtain

$$y^3 + Py + Q = 0$$

where

$$P = \frac{-b^2}{3a^2} + \frac{c}{a} \text{ and } Q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$$

Now, we have reduced the problem of solving the original cubic equation. If we set $y = z - \frac{P}{3z}$ and work out the algebra, we get

$$z^6 + Qz^3 - \frac{P^3}{27} = 0$$

The above polynomial is quadratic in z^3 thus, z^3 can be found using the quadratic formula

$$z^3 = \frac{-Q}{2} \pm \sqrt{\Delta} \text{ where } \Delta = \frac{Q^2}{4} + \frac{P^3}{27}.$$

Where Δ is the discriminant of the cubic equation. We can take the cube root to get $z = \sqrt[3]{\frac{-Q}{2} \pm \sqrt{\Delta}}$

Now, together with the equations

$$y = z - \frac{P}{3z} \text{ and } X = y - \frac{b}{3a}$$

We have the cubic formula. □

This formula allows us to compute the solution for X of an original cubic equation. The equation we have for z gives three complex cube roots for each of the positive and negative signs, hence 6 different formulas for z. After substituting these equations into the equation for y, we get at most 3 different resulting y values [6]. Thus, the last equation will give at most 3 distinct roots x.

Theorem 4.3. *Solution to the biquadratic(quartic) equation.*

Proof. Consider $aX^4 + bX^3 + cX^2 + dX + e = 0$ where a,b,c,d, and e are real numbers such that $a \neq 0$. First, divide by a to get

$$X^4 + \frac{b}{a}X^3 + \frac{c}{a}X^2 + \frac{d}{a}X + \frac{e}{a} = 0$$

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Now, we can substitute $X = y - \frac{b}{4a}$. After some algebra, we get an equation of the form

$$y^4 + py^2 + qy + r = 0$$

where p,q and r are new coefficients that can be computed from the previous ones. We can write this as $y^4 = -py^2 - qy - r$. We can then manipulate this equation using a value, say z, that we will determine later:

$$y^4 + 2y^2z^2 + z^4 = -py^2 - qy - r + 2y^2z^2 + z^4$$

$$(y^2 + z^2)^2 = (2z^2 - p)y^2 - qy + (z^4 - r)$$

The value for z which we will pick will make the right hand side of the equation equal to $(fy + g)^2$ for some particular values f and g. Then we will have

$$(y^2 + z^2)^2 = (fy + g)^2$$

$$y^2 + z^2 = \pm(fy + g)$$

$$\begin{cases} \mathrm{y}^2 - fy + (z^2 - g) = 0 \\ y^2 + fy + (z^2 + g) = 0 \end{cases}$$

Now, we have to find z, f and g such that

$$(2z - p)y^2 - qy + (z^4 - r) = (fy + g)^2$$

We can consider the equations formed by setting each side of this equation equal to 0. The right hand side, $(fy + g)^2 = 0$, would have only one root which is $y = \frac{-g}{f}$. Then, the left hand side of the equation,$(2z - p)y^2 - qy + (z^4 - r) = 0$, will also have only one root. This equation may also be solved for y by using the quadratic formula, and in order for this formula to yield only one root, the discriminant must be 0. That is, we can find z, f, and g just when

$$q^2 - 4(2z^2 - p)(z^4 - r) = 0$$

$$-8z^6 + 4pz^4 + 8rz^2 + (q^2 - 4pr) = 0$$

$$\begin{cases} \text{w=z}^2 \\ 8w - 4pw^2 - 8rw + (4pr - q^2) = 0 \end{cases}$$

The last equation can be solved using the cubic formula, resulting in at most 3 solutions, $w = w_0, w_1, w_2$. For each j the equation

$$(2w_j - p)y^2 - qy + (w_j^2 - r) = 0$$

has a single root

$$y = \frac{q}{2(2w_j - p)}$$

Therefore,

$$(2w_j - p)y^2 - qy + (w_j^2 - r) = (2w_j - p)(y - \frac{q}{2(2w_j - p)})^2$$

$$(y\sqrt{2w_j - p} - \frac{q}{2\sqrt{2w_j - p}})^2 = (fy + g)^2$$

Where $f = \sqrt{2w_j - p}$ and $g = \frac{-q}{2f}$

With these three values of f and g, solve the two quadratics

$$\begin{cases} \text{y}^2 - fy + (z^2 - g) = 0 \\ y^2 + fy + (z^2 + g) = 0 \end{cases}$$

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for y. This results in as many as 12 possible y formulas, but at most four will have distinct values. Finally, calculate the roots

$$X = y - \frac{b}{4a}$$

□

5 Implications for Teaching

When it comes to teaching, incorporating the history of algebra into the curriculum can have a positive impact in the classroom. When formulas and other types of mathematical expressions are first presented, they can seem very intimidating at first glance. In order to decrease this intimidation factor, more context can be given such as proofs and history of mathematics. Some of the positive impacts are improving students problem solving skills, providing a better foundation for future understanding, helping students make better mathematical connections, and highlighting the interaction between mathematics and real life situations. Not only can incorporating history in teaching mathematics result in these positive impacts, but it can also help students increase their motivation to learn. It can humanize mathematics for students and help them to realize that even famous mathematicians struggled at times.

One study was done to investigate the impacts of implementing a historical approach through the teaching and learning of the topic of the integral of secant, a topic commonly taught in the second semester calculus courses at both high school and university level [9]. This study involved 16 undergraduate students who were mathematics majors. There were 9 males and 7 females selected who had all completed their calculus sequence. These students were given a "take home" document one week before being asked to complete an "in class" document. The first "take home" document consisted of background knowledge of the topic and the historical need for it. Key historical figures were discussed as well as the important role the topic played. The specific topic in this study was the Mercator projection. The second document or "in class" portion had students explore the integral of secant. In this document, the intention was to lead the students through a historical reenactment of this integral's discovery which was motivated by a desire for a mathematical description of the Mercator projection. Within this document, students were asked to complete a variety of tasks. One of these tasks included filling in the blanks to finish traditional proofs. Each student was asked the following questions:

1. Describe what you learned in the activity on the historical approach to the integral of the secant.

2. How was the activity different from a typical mathematics class?

3. Here is the calculus textbook that we use here at the University of Montana. This is the presentation for the integral of the secant. How does it differ from the historical presentation of the integral of the secant that was presented last week?

4. Did the activity change the way that you view mathematical discovery?

5. Did the activity change the way that you view learning mathematics?

6. Would you say that you were more or less motivated to complete the traditional proof of the integral of the secant after having placed its discovery in a historical context

7. Does including mathematics history make mathematics more meaningful? How? [9]

The student's responses to these questions were audio recorded and transcribed for analysis. While this new approach was implemented, the students displayed intense curiosity in mathematics. Question 5 refers to how the implementation of history and proofs in math classrooms impacted the way in which students viewed learning mathematics. The answer to this question can be essential in discovering what works for the students and what doesn't. Below is a response that one of the students had to question 5:

I found it that it made me feel that my work was more important than it usually is. And the fact that usually when you do a problem, you get an answer, and

> think you're done, but, there really is no point to it that you see... I mean... when you're taking... you're doing integration by parts, it's like, okay, what are we ever going to use this for? And so, you do all this work and you never see really ever where it applies... they'll try to do stuff and... I mean it's really, really basic and it doesn't really apply, but, if they could take examples and show where its used,the historical context, it makes it feel as if you're kind of working along side of those people when they were actually doing the work hundreds of years ago... you went through and saw what they did, and so it gives a level of importance that isn't usually ever there... you know... that was valuable. (Student 1)

This specific response displays many positive signs. The student emphasizes how they felt that the work they were doing had more importance, which can improve motivation for the subject. Taking into consideration all the other students responses, the analysis of these responses was unanimous in the education techniques approval. Some of the reasons against this technique may be that the history will distract from the course material or that the students will find it boring. This student's response contradicts both of these reasons. These students were more motivated and engaged by this technique. They seemed to have a much better grasp of the material at hand.

Another study took place in Singapore. This study used an action-research approach. A sample of 102 students in Singapore Polytechnic who enrolled for the Certificate of Engineering Mathematics (CEM) took part in the study [10]. A similar type of historical approach, like the one above, was applied to a linear algebra course that was taught by the author. The existing course syllabus was used, but the history of mathematics was strategically integrated. They were able to teach the entire syllabus like usual without any interference to the time schedule. Each student was asked to keep a log in order to write their thoughts and feelings about each lesson for the duration of the course which was 12 weeks. Not only did the students keep a log, but the teacher did as well. The purpose of this was for them to keep track of the teaching and learning processes. These logs would provide qualitative information on the motivation aspects of both the students and the instructor.

The results of this study was quite promising. There were many positive effects to the historical approach shown in the students' logs. It seemed to positively impact the attitudes that the students had towards the subject. Many students reported in their logs how motivated they were after the lessons. They were able to take more out of the lesson than just the formulas and computations. Overall this study shows how the history of mathematics possesses a strong relationship with the subject of mathematics.

Along with these benefits, there are definitely still obstacles to overcome with this implementation. It takes extra effort on the teachers end which can make things difficult.The teachers must still be able to get through all the material they need to cover. It takes extra time to cover some of the history of mathematics within a course. Teachers have also expressed concerns that incorporating too much of the history may distract from the basic mastery of mathematical skills. Due to this fact, many teachers are very resistant to this type of approach. Also, there is a common argument against the use of history in mathematics courses. This argument is that students typically find it boring. The results of both of these studies appears to contradict that argument as the students expressed their increased motivated and engagement with the material.

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