Exploration of Patterns in Triangular Arrays

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MA 398: Senior Seminar

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Fall 2021

Abstract

Triangular arrays are a method for arranging coefficients of polynomials in a way that allows mathematicians to find and prove patterns within them. While some arrays, like Pascal's Triangle, are widely known and explored, other triangular arrays, like the Factorial Triangle and Euler's Number Triangle, are less known. We will explore the patterns within the aforementioned lesser known arrays.

1 Introduction

Triangular arrays are used by algebraists to model the coefficients of polynomials. Of the triangular arrays that have been studied, perhaps the most popular is Pascal's triangle. Despite the name, Blaise Pascal, after whom the triangle is named after, was not the first mathematician to arrange numbers in the order they are found in the triangle. The Chinese algebraist Chu Shï-kié was the first to arrange the coefficients in the 1300s [6]. However, Pascal's triangle wasn't officially published until 1527 when it appeared in *The Arithmetic of Apianus*, a book by an Petrus Apianus [6]. The array was published and explored by several other mathematicians before Pascal published his findings (posthumously). When Pascal published the array, it appeared in its familliar form, which is different than previous examples of it, along side the many patterns found in the triangle [1]. One of the more important ideas in this book was the Binomial Theorem we currently use for positive integral exponents [1]. This theorem helps us expand binomials of the form $(a + b)^n$ where n corresponds to the row number of Pascal's triangle (where the number of the first row equals 0) [1]. For example, $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ where the coefficients of the variables are the entries to the fourth row of Pascal's triangle [1]. If we were given the equation $(x + 2)^3$, we could use this theorem to quickly expand it to $x^3 + 3(2)x^2 + 3(2^2)x + 2^3$ by setting $a = x$ and $b = 2$. This is just one example of how triangular arrays may help in solving polynomials.

Asides from Pascal's Triangle, there are other triangular arrays that have been discovered and explored throughout history. Among these are Lucas Triangle, Bell's Triangle, the Factorial Triangle, and Euler's Number Triangle. In this paper we will explore the patterns present in the Factorial Triangle and Euler's Number Triangle. Interestingly, these triangles can be modeled by more than one equation. Each array has a recursive equation and an equation that relates to the corresponding polynomial. It is from the equation that connects the entries of the arrays to their respective polynomial that these arrays derive their importance. Instead of having to calculate out complicated polynomials, one could simply use the entries of the array (similar to how Pascal's Triangle can be used for binomial expansion). In this paper we will use the recursive equation for each array as its definition and then prove its relation to the secondary equation.

2 Preliminaries

2.1 Triangular Arrays

Triangular arrays are arrays of variables that are double-indexed. First they are indexed by the row, then by the number of places from the left. Thus, triangular arrays follow the form

$$
\begin{array}{ccccc}
 & & X_{1,1} & & \\
 & X_{2,1} & & X_{2,2} & \\
X_{3,1} & & X_{3,2} & & X_{3,3} \\
 & & \vdots & & \\
X_{i,1} \quad X_{i,2} & & \ldots & & X_{i,i}
\end{array}
$$

where $i \in \mathbb{N}$ and $i \geq 1$. We know that i must be a natural number because we are dealing with places, which are only positive counting numbers [5].

2.2 Naming

For each array, we will maintain a consistent naming pattern. Each entry in the Factorial Triangle will be denoted as $A_{i,j}$ and $B_{i,j}$ for Euler's Number Triangle, where i denotes the row number and j denotes the number of entries from the left. Additionally, due to the triangular shape of the array, each row number also denotes the number of possible entries in each row. Thus we will be using $i = 1$ as the first value of i since the first row must have at least one entry. Hence, $i \geq 1$. Similarly, $j \geq 1$. Along these same lines, $j \leq i$, since we cannot have more entries from the left than there are entries in a row.

These observations result in the following equations for both triangular arrays:

$$X_{i,j} = 0 \text{ if } i < 1 \tag{1}$$

$$X_{i,j} = 0 \text{ if } j < 1 \tag{2}$$

$$X_{i,j} = 0 \text{ if } j > i \tag{3}$$

3 The Factorial Triangle

The first triangular array that we will look at is the Factorial Triangle. This triangle was discovered by Steven Schwartzman in the 1980s. He based his work on the then recent work of Kenneth Kundert who had discussed synthetic multiplication for $(a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0)(x + k)$ [4]. The first five rows of this array are below.

1

1 1

1 3 2

1 6 11 6

1 10 35 50 24

We will define this triangle by the recursive equation:

$$A_{i,j} = [A_{i-1,j-1} * (i-1)] + A_{i-1,j}, \text{ where } A_{1,1} = 1\ .$$

3.1 Specific Entries

There are several patterns in this triangle regarding specific entries of each row. These entries can be found using simpler equations rather than the equation used to define the whole array. In other words, for these entries secondary, non-recursive equations can be used, but these equations do not necessarily work for other entries.

A noticeable pattern of the Factorial Triangle is that the first entry of every row is one, which we will prove here.

Lemma 3.1.1 $A_{i,1} = 1$ for all $i \geq 1$.

Proof. We will prove this directly. Let us consider $A_{i,1}$, $i \geq 1$.

$$\begin{aligned} A_{i,1} &= [A_{i-1,0} * (i-1)] + A_{i-1,1} \\ &= [0 * (i-1)] + A_{i-1,1} \qquad A_{i-1,0} = 0 \text{ by Equation 2} \\ &= A_{i-1,1} \end{aligned}$$

Since this is true for all $i \geq 1$, we find

$$A_{i,1} = A_{i-1,1} = A_{i-2,1} = ... = A_{1,1} = 1 \ .$$

Hence, our desired result. □

Another pattern that can be discerned in the Factorial Triangle is the second entry of every row can be found by adding the previous row's second entry with the previous entry's row number. In other words, $A_{i,2} = A_{i-1,2} + (i-1)$.

Lemma 3.1.2 $A_{i,2} = A_{i-1,2} + (i-1)$ for all $i \geq 1$.

Proof. Let us consider $A_{i,2}$.

$$\begin{aligned} A_{i,2} &= [(A_{i-1,2-1} * (i-1)] + A_{i-1,2} \\ &= (i-1)A_{i-1,1} + A_{i-1,2} \\ &= (i-1)(1) + A_{i-1,2} && A_{i-1,1} = 1 \text{ by Lemma 3.1.1} \\ A_{i,2} &= A_{i-1,2} + (i-1) \end{aligned}$$

Hence our desired result. □

Example 3.1.3 Let us consider $A_{3,2}$ as an example of this Lemma. Thus we have

$$\begin{aligned} A_{3,2} &= A_{2,2} + (3-1) && \quad\quad 1 \\ &= 1 + 2 && \quad 1 \quad 1 \\ &= 3 && 1 \quad \mathbf{3} \quad 2 \end{aligned}$$

Observe that the answer we got from Lemma 3.1.2 is matches the second entry of the third row (in bold).

Another noticeable pattern in the Factorial Triangle is that the last entry of every row is equal to the factorial of the previous row, which we will prove here.

Lemma 3.1.4 $A_{i,i} = (i-1)!$ for all $i \geq 1$.

Proof. We will proceed with proof by induction.

Base Case: By definition $A_{1,1} = 1$ and we see that $(1-1)! = 0! = 1$.

Induction Hypothesis: Let $k \in \mathbb{N}$, $k > 0$ and suppose $A_{k,k} = (k-1)!$. We will show that $A_{(k+1),(k+1)} = k!$

$$\begin{aligned} A_{(k+1),(k+1)} &= [A_{(k+1)-1,(k+1)-1} * ((k+1)-1)] + A_{(k+1)-1,k+1} \\ &= [A_{k,k} * (k)] + A_{k,k+1} \qquad A_{k,k} = (k-1)! \text{ by the Induction Hypothesis} \\ &= [(k-1)! * k] + 0 \qquad A_{k,k+1} = 0 \text{ by Equation 3} \\ &= k! \end{aligned}$$

Thus we have shown that if $A_{k,k} = (k-1)!$ then $A_{(k+1),(k+1)} = k!$

Hence our desired result. □

Example 3.1.5 Let us consider $A_{4,4}$ as an example for this Lemma. Thus we have

$$\begin{aligned} A_{4,4} &= (4-1)! \\ &= 3! \\ &= 6 \end{aligned}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 2 & \\ 1 & & 6 & & 11 & & \mathbf{6} \end{array}$$

Observe that the answer we got from Lemma 3.1.3 is matches the last entry of the fourth row (in bold).

3.2 Another Equation for the Factorial Triangle

In addition to the equation used to define this triangular array, there is another equation which can model the entries of this array,

$$\prod_{n=0}^{i-1}(x+n) = A_{i,1} * x^i + A_{i,2} * x^{i-1} + ... + A_{i,i-1} * x^2 + A_{i,i} * x \text{ where } i \in \mathbb{N}\ .$$

In this instance, the coefficients of the resulting polynomial are the corresponding entries to the array.

Example 3.2.1 For example, let us consider $i = 3$. By the purposed equation we see

$$\begin{aligned}\textstyle\prod_{n=0}^{2}(x+n) &= (x+0)(x+1)(x+2)\\ &= x^3 + 3x^2 + 2x\end{aligned}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & \mathbf{1} & & \mathbf{3} & & \mathbf{2} & \\ 1 & & 6 & & 11 & & 6 \end{array}$$

Note the coefficients in the above polynomial correspond to the third row of the Factorial Triangle, as indicated in bold. We will investigate this equation for all natural numbers $i \geq 1$.

Theorem 3.2.2:

$$\prod_{n=0}^{i-1}(x+n) = A_{i,1} * x^i + A_{i,2} * x^{i-1} + ... + A_{i,i-1} * x^2 + A_{i,i} * x$$

for all $i \geq 1$ where $A_{i,j}$ are the respective entries of the Factorial Triangle.

Proof. : We will proceed with proof by induction.

Base Case: By definition $A_{1,1} = 1$. Using the proposed equation we find $(x+0) = x$ where x has a coefficient of one. Thus our base case is true.

Induction Hypothesis: Let $k \in \mathbb{N}$, $k > 0$ and suppose

$$\prod_{n=0}^{k-1}(x+n) = A_{k,1} * x^k + A_{k,2} * x^{k-1} + ... + A_{k,k-1} * x^2 + A_{k,k} * x \ .$$

We will show that

$$\prod_{n=0}^{k}(x+n) = A_{k+1,1} * x^{k+1} + A_{k+1,2} * x^k + ... + A_{k+1,k} * x^2 + A_{k+1,k+1} * x \ .$$

Observe that

$$\prod_{n=0}^{k}(x+n) = \left[\prod_{n=0}^{k-1}(x+n)\right] * (x+k) \ .$$

Thus, by the induction hypothesis,

$$\begin{aligned}
\textstyle\prod_{n=0}^{k}(x+n) &= [A_{k,1} * x^k + A_{k,2} * x^{k-1} + A_{k,3} * x^{k-2} + ... + A_{k,k-2} * x^3 + A_{k,k-1} * x^2 \\
&\quad + A_{k,k} * x] * (x+k) \\
&= A_{k,1} * x^{k+1} + kA_{k,1} * x^k + A_{k,2} * x^k + kA_{k,2} * x^{k-1} + A_{k,3} * x^{k-1} \\
&\quad + kA_{k,3} * x^{k-2} + ... + A_{k,k-2} * x^4 + kA_{k,k-2} * x^3 + A_{k,k-1} * x^3 \\
&\quad + kA_{k,k-1} * x^2 + A_{k,k} * x^2 + kA_{k,k} * x \\
&= A_{k,1} * x^{k+1} + (kA_{k,1} + A_{k,2}) * x^k + (kA_{k,2} + A_{k,3}) * x^{k-1} \\
&\quad + (kA_{k,3} + A_{k,4}) * x^{k-2} + ... + (kA_{k,k-3} + A_{k,k-2}) * x^4 \\
&\quad + (kA_{k,k-2} + A_{k,k-1}) * x^3 + (kA_{k,k-1} + A_{k,k}) * x^2 + kA_{k,k} * x \ .
\end{aligned}$$

Notice that $(kA_{k,j-1} + A_{k,j})x^{k-j+2} = A_{k+1,j}(x^{k-j+2})$ for each $2 \leq j \leq k$. This is true since

$$\begin{aligned} A_{k+1,j} &= [(A_{(k+1)-1,j-1} * ((k+1)-1)] + A_{(k+1)-1,j} \\ &= A_{k,j-1}(k) + A_{k,j} \ . \end{aligned}$$

This equation also applies for the coefficients of the x^{k+1} and x terms (which are special cases due to having only one term). For $A_{k,1}x^{k+1}$ we see that the coefficient is referring to the first term of the k^{th} row. By Lemma 3.1.1, we know that $A_{k,1} = A_{k+1,1} = 1$. Thus, $A_{k,1}x^{k+1} = A_{k+1,1}x^{k+1}$. For the last term, $(kA_{k,k}) * x$, we will show that $A_{k+1,k+1} = kA_{k,k}$.

$$\begin{aligned} A_{k+1,k+1} &= [A_{(k+1)-1,(k+1)-1} * ((k+1)-1] + A_{(k+1)-1,k+1} \\ &= A_{k,k} * (k) + A_{k,k+1} \qquad A_{k,k+1} = 0 \text{ by Equation 3.} \\ &= kA_{k,k} \end{aligned}$$

We can replace the coefficients with the corresponding $A_{k+1,j}$ values, yielding

$$\begin{aligned} \prod_{n=0}^{k}(x+n) &= A_{k+1,1} * x^{k+1} + A_{k+1,2} * x^{k} + A_{k+1,3} * x^{k-1} + A_{k+1,4} * x^{k-2} \\ &+ ... + A_{k+1,k-2} * x^4 + A_{k+1,k-1} * x^3 + A_{k+1,k} * x^2 + A_{k+1,k+1} * x \ . \end{aligned}$$

Note this is an expanded definition of $\prod_{n=0}^{k}(x+n)$ that we were trying to prove. Hence our desired result. □

4 Euler's Number Triangle

The next triangular array that we will look at is Euler's number triangle. This triangle consists of the Eulerian Numbers from which the coefficients of the

Eulerian Polynomials are derived from (the Eulerian Numbers are only part of the coefficient). Although interesting, we will not be exploring Eulerian polynomials within the scope of this paper (see [2]). Instead, we will look at the relationship between the entries of the triangle and Eulerian Numbers which we will define in Section 4.2. Before looking at Eulerian Numbers, we will look at patterns contained within the triangle. The first five rows of this array are below.

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

We will define this triangle by the recursive equation

$$B_{i,j} = [B_{i-1,j-1} * (i - j + 1)] + B_{i-1,j} * (j) \text{ where } B_{1,1} = 1 \text{ .}$$

4.1 Specific Entries

Similar to the patterns found in the Factorial Triangle discussed in the previous section, there are several patterns that can be noticed in Euler's number triangle. These patterns also remain specific to certain entries. For example, just by looking at the triangular array, we notice that it appears that the first and last entry of every row is one. However, to prove that this pattern continues for all rows, we will consider each pattern independently. We also notice that each row is symmetric.

The first pattern we will prove is that the initial entry of each row of the array will be one. As a formula, we will show that $B_{i,1} = 1$ for all $i \geq 1$.

Lemma 4.1.1 $B_{i,1} = 1$ for all $1 \geq 1$.

Proof. We will prove this directly. Let us consider $B_{i,1}$, $i \geq 1$.

$$\begin{aligned} B_{i,1} &= [B_{i-1,0} * (i)] + B_{i-1,1} * (1) \\ &= [0] + B_{i-1,1} \\ &= B_{i-1,1} \end{aligned}$$

Since this is true for all $i \geq 1$, we find

$$B_{i,1} = B_{i-1,1} = B_{i-2,1} = ... = B_{1,1} = 1 \ .$$

Hence our desired result. □

Now we will prove that each row is symmetric.

Theorem 4.1.2: $B_{i,j} = B_{i,i-j+1}$ for all $i, j \geq 1$.

Proof. We will proceed with proof by induction.

Base Case: The first row is trivial because there is only one entry. Thus, we will investigate and check that $B_{2,1} = B_{2,(2-1+1)} = B_{2,2}$. By Lemma 4.1.1 $B_{2,1}$. Let us consider $B_{2,2}$.

$$\begin{aligned}
B_{2,2} &= (B_{2-1,2-1} * (2-2+1)) + B_{2-1,2} * (2) \\
&= (B_{1,1} * (1)) + B_{1,2} \qquad\qquad B_{1,2} = 0 \text{ by Equation 3} \\
&= (1 * 1) + (0 * 2) \\
B_{2,2} &= 1
\end{aligned}$$

Since $B_{2,1} = B_{2,(2-1+1)} = B_{2,2}$, our base case is true.

Induction Hypothesis: Let $k \in \mathbb{N}$, $k > 0$ and suppose $B_{k,j} = B_{k,k-j+1}$. We will show that $B_{k+1,j} = B_{k+1,(k+1)-j+1} = B_{k+1,k-j+2}$.

First we evaluate $B_{k+1,j}$

$$\begin{aligned}
B_{k+1,j} &= (B_{(k+1)-1,j-1} * ((k+1)-j+1)) + B_{k+1-1,j} * j \\
&= (k-j+2) * B_{k,j-1} + j * B_{k,j}
\end{aligned}$$

Then we will evaluate $B_{k+1,k-j+2}$.

$$\begin{aligned}
B_{k+1,k-j+2} &= [B_{(k+1)-1,(k-j+2)-1)} * ((k+1)-(k-j+2)+1)] \\
&\quad + B_{(k+1)-1,k-j+2} * (k-j+2) \\
&= j * B_{k,k-j+1} + (k-j+2) * B_{k,k-j+2} \\
B_{k,k-j+1} = B_{k,j} &\quad \text{by the Induction Hypothesis} \\
&= j * B_{k,j} + (k-j+2) * B_{k,k-j+2} \\
B_{k,j-1} = B_{k,k-j+2} &\quad \text{by the Induction Hypothesis} \\
&= j * B_{k,j} + (k-j+2) * B_{k,j-1}
\end{aligned}$$

We note that this last line is equal to our expression for $B_{k+1,j}$ above. Hence our desired result. □

Combining Lemma 4.1.1 and Theorem 4.1.2, we notice that the last entry in every row is one, giving us the next corollary.

Corollary 4.1.3: $B_{i,i} = 1$ for all $i \geq 1$.

Proof. From Lemma 4.1.1, we know that $B_{i,1} = 1$ for all $i \geq 1$. Theorem 4.1.2 shows that $B_{i,i} = B_{i,i-i+1} = B_{i,1}$. Thus, $B_{i,i} = B_{i,1} = 1$ for all $i \geq 1$. □

4.2 Generalized Patterns

Aside from the patterns relating to specific entries discussed in the previous section, Euler's Number Triangle also has other patterns that are more generalized. This includes a pattern regarding the sum of all the entries in a row as well as another equation which models the entries of Euler's Number Triangle.

The first of these broader patterns that we will investigate is that the sum of all of the entries of a row equals the factorial of the row. In other words, if we add up all the entries of a row, it is equal to the factorial of the row number.

Example 4.2.1 Let us consider the fourth row as an example of this proposed equation. Thus we have

$$\begin{aligned}\sum_{j=1}^{4} B_{4,j} &= B_{4,1} + B_{4,2} + B_{4,3} + B_{4,4} \\ &= 1 + 11 + 11 + 1 \\ &= 24 = 4!\end{aligned}$$

Observe that the sum of the entries of the fourth row of Euler's Number Triangle equals 4! which is our desired result.

Theorem 4.2.2 $\sum_{j=1}^{i} B_{i,j} = B_{i,1} + B_{i,2} + ... + B_{i,i} = i!$.

Proof. We will proceed with proof by induction.

Base Case: By definition $B_{1,1} = 1$. We know that $1! = 1$, hence the pattern holds for the first row.

Induction Hypothesis: Let $k \in \mathbb{N}$, $k > 0$ and suppose

$$\sum_{j=1}^{k} B_{k,j} = B_{k,1} + B_{k,2} + ... + B_{k,k} = k!$$

We will show

$$\sum_{j=1}^{(k+1)} B_{k+1,j} = B_{k+1,1} + B_{k+1,2} + ... + B_{k+1,k} + B_{k+1} = (k+1)!\ .$$

We start with the equation

$$\sum_{j=1}^{(k+1)} B_{k+1,j} = B_{k+1,1} + B_{k+1,2} + ... + B_{k+1,k} + B_{k+1,k+1}$$

Recall that our recursive formula says

$$\begin{aligned} B_{k+1,j} &= [B_{(k+1)-1,j-1} * ((k+1) - j + 1)] + B_{(k+1)-1,j} * (j) \\ &= [B_{k,j-1} * (k - j + 2)] + (j)B_{k,j}\ . \end{aligned}$$

Using this general equation, we will substitute the values of $B_{k+1,j}$ in our original summation equation.

$$\sum_{j=1}^{(k+1)} B_{k+1,j} = B_{k,0}(k+1) + (1)B_{k,1} + kB_{k,1} + 2B_{k,2} + B_{k,2}(k-1) + 3B_{k,3}$$
$$+ ... + 3B_{k,k-2} + (k-1)B_{k,k-1} + 2B_{k,k-1} + kB_{k,k}$$
$$+ B_{k,k} + (k+1)B_{k,k+1}$$

$B_{k,0} = 0$ and $B_{k,k+1} = 0$ because of Equations 2 and 3, respectively. Thus our equation becomes

$$= (k+1)B_{k,1} + (2 + (k-1))B_{k,2} + ... + ((k-1)+2)B_{k,k-1}$$
$$+ (k+1)B_{k,k}$$
$$= (k+1)B_{k,1} + (k+1)B_{k,2} + ... + (k+1)B_{k,k-1} + (k+1)B_{k,k}$$
$$= (k+1) * (B_{k,1} + B_{k,2} + ... + B_{k,k-1} + B_{k,k})$$
$$= (k+1) * k! \qquad \text{by the Induction Hypothesis}$$
$$= (k+1)!$$

Thus we have shown if $\sum_{j=1}^{k} B_{k,j} = k!$ then $\sum_{j=1}^{(k+1)} B_{k+1,j} = (k+1)!$. Hence our desired result. □

Similar to the Factorial Triangle, there is another equation that models the entries of this triangular array. The other equation that models the entries of this triangular array is a summation equation which produces Euler's numbers. Euler's numbers count the number of permutations of length i with $j-1$ accents [7]. Permutations refer to the ways which i numbers can be ordered, while $j-1$

accents refers to the number of occurrences where the next number is greater than the previous number [8]. The number of permutations with i digits and $j-1$ accents is

$$\sum_{k=0}^{j-1}(-1)^k * \binom{i+1}{k} * (j-k)^i \text{ . [7]}$$

We will prove that the $B_{i,j}$ entry of the triangle equals the value of the summation equation. Before we do that, let's look at an example.

Example 4.2.3 For example, consider the possible permutations of length 3.

Permutation	Number of accents
$(1, 2, 3)$	2
$(1, 3, 2)$	1
$(2, 3, 1)$	1
$(2, 1, 3)$	1
$(3, 1, 2)$	1
$(3, 2, 1)$	0

Note above that there is one permutation with two accents, four permutations with one accent, and one permutation with no accents. Using the notation used for our array (noting that the number of permutations must be shifted by +1 to account for our naming conventions) we see the above results can be written as

		1
$B_{3,1}$	$= 1$	1 1
$B_{3,2}$	$= 4$	**1 4 1**
$B_{3,3}$	$= 1$	1 11 11 1

These results correspond to the third row of Euler's Number Triangle, as noted in bold above. Furthermore we see this matches the summation as

$$B_{3,1} = \tbinom{4}{0}1^3 = 1 \qquad B_{3,2} = \tbinom{4}{0}2^3 - \tbinom{4}{1}1^3 = 8 - 4 = 4 \qquad B_{3,3} = \tbinom{4}{0}3^3 - \tbinom{4}{1}2^3 + \tbinom{4}{2}1^3 = 27 - 32 + 6 = 1$$

Before we prove that the entries in Euler's Triangle match Euler's Numbers, let's recall the combinatorical result [8] below:

$$\binom{j}{k} + \binom{j}{k+1} = \binom{j+1}{k+1}. \tag{4}$$

We will also use a summation result from Ruiz's paper [3]. His Corollary 2 states: For all integers $i \geq 0$ and for all real numbers x

$$\sum_{k=0}^{i}(-1)^k * \binom{i}{k} * (x-k)^{i-j} = 0 \qquad 1 \leq j \leq i\ .$$

We can adjust this to match our needs by replacing i with $i+1$, setting $j = 1$, and defining $x = i$. This results in the following equation which we will use in our next proof.

$$\sum_{k=0}^{i+1}(-1)^k * \binom{i+1}{k} * (i-k)^i = 0\ . \tag{5}$$

Theorem 4.2.4:

$$B_{i,j} = \sum_{k=0}^{j-1} (-1)^k * \binom{i+1}{k} * (j-k)^i \quad \text{for all } i, j \geq 1 \ .$$

Proof. We will proceed with proof by induction where induction is performed on the diagonals of Euler's triangle which are formed by the entries with the same j values. In our induction hypothesis we will assume its true for the j^{th} diagonal then prove it is true for the $(j+1)^{st}$ diagonal. To do so, we need to know that it is true for the first element in the $(j+1)^{st}$ diagonal. Then we will prove that if it is true for some element in the $(j+1)^{st}$ diagonal (say the m^{th} row), then it will be true for the next element in the diagonal (which would be in the $(m+1)^{st}$ row). Thus for our base case we need to show it is true for the first diagonal, $B_{i,1}$ for all i, and also that it is true for the first entry of the remaining diagonals (the last entry of every row), $B_{i,i}$ for all i.

Base Case: For our base case, we will consider two values, $B_{i,1}$ and $B_{i,i}$, since both are required by our induction hypothesis. By Euler's Number Triangle, both of these values should equal one. We will first consider $B_{i,1}$.

$$\begin{aligned}
\textstyle\sum_{k=0}^{1-1}(-1)^k * \binom{i+1}{k} * (1-k)^i \quad &= \textstyle\sum_{k=0}^{0}(-1)^k * \binom{i+1}{k} * (1-k)^i \\
&= (-1)^0 * \tbinom{i+1}{0} * (1-0)^i \\
\tbinom{i+1}{0} = 1 \quad & \text{by combinatoric rules and } 1^i = 1 \text{ for all } i \geq 1 \\
&= 1 * 1 * 1 \\
&= 1 = B_{i,1} \qquad \text{by Lemma 4.1.1}
\end{aligned}$$

Thus $\sum_{k=0}^{0}(-1)^k * \binom{i+1}{k} * (1-k)^i = B_{i,1}$. Therefore this base case is true.

Next we will consider the summation equation for $B_{i,i}$ and show that it equals $B_{i,i} = 1$. We will do this by initially considering the sum to $i+1$, $\sum_{k=0}^{i+1}(-1)^k * \binom{i+1}{k} * (i-k)^i$. By Equation 5 above we know that this sum equals 0. Now we can manipulate the sum to get

$$\begin{aligned}
0 = \sum_{k=0}^{i+1}(-1)^k\binom{i+1}{k}(i-k)^i &= \left[\sum_{k=0}^{i-1}(-1)^k\binom{i+1}{k}(i-k)^i\right] + (-1)^i\binom{i+1}{i}(i-i)^i \\
&\quad + (-1)^{i+1}\binom{i+1}{i+1}(i-(i+1)^i \\
&= \left[\sum_{k=0}^{i-1}(-1)^k\binom{i+1}{k}(i-k)^i\right] + (-1)^i\binom{i+1}{i} * 0^i \\
&\quad + (-1)^{i+1}(1)(-1)^i \\
\binom{i+1}{i+1} = 1 \text{ by combinatoric } &\text{ rules} \\
&= \left[\sum_{k=0}^{i-1}(-1)^k\binom{i+1}{k}(i-k)^i\right] + 0 + (-1)^{2i+1} \\
(-1)^{2i+1} = -1 \text{ for all } i \geq 1 \quad &\text{ because } (2i+1) \text{ will always be odd} \\
&= \sum_{k=0}^{i-1}(-1)^k\binom{i+1}{k}(i-k)^i - 1 \\
1 &= \sum_{k=0}^{i-1}(-1)^k\binom{i+1}{k}(i-k)^i
\end{aligned}$$

Therefore $\sum_{k=0}^{i-1}(-1)^k\binom{i+1}{k}(i-k)^i = 1 = B_{i,i}$ by Corollary 4.1.3. Thus, both of our base cases are true.

Induction Hypothesis: We will perform induction on the diagonals of Euler's Triangle. Thus we will assume that it is true on the j^{th} diagonal and show it is true for the $(j+1)^{st}$ diagonal. Let $i, j \in \mathbb{N}$, and suppose

$$B_{i,j} = \sum_{k=0}^{j-1}(-1)^k * \binom{i+1}{k} * (j-k)^i \text{ for all} i \geq j \ .$$

To prove it is true for the $(j+1)^{st}$ diagonal we will also assume the equation holds for the m^{th} row of the $(j+1)^{st}$ diagonal where $m \geq j+1$. Thus we will also suppose that

$$B_{m,j+1} = \sum_{k=0}^{j} (-1)^k * \binom{m+1}{k} * (j+1-k)^m \; .$$

We will now show the equation must hold for the remaining terms in the $(j+1)^{st}$ diagonal. Thus we will show

$$B_{m+1,j+1} = \sum_{k=0}^{j} (-1)^k \binom{m+2}{k} (j+1-k)^{m+1} \; .$$

$$\begin{aligned} B_{m+1,j+1} &= [B_{(m+1)-1,(j+1)-1} * ((m+1)-(j+1)+1)] + B_{(m+1)-1,j+1}(j+1) \\ &= (m-j+1)B_{m,j} + (j+1)B_{m,j+1} \end{aligned}$$

By the two Induction Hypotheses we have

$$\begin{aligned} &= \left[\textstyle\sum_{k=0}^{j-1} (-1)^k * \binom{m+1}{k} * (j-k)^m\right](m-j+1) \\ &\quad + \left[\textstyle\sum_{k=0}^{j} (-1)^k * \binom{m+1}{k} * (j+1-k)^m\right](j+1) \\ &= [\tbinom{m+1}{0}(j^m) - \tbinom{m+1}{1}(j-1)^m + \tbinom{m+1}{2}(j-2)^m \\ &\quad - ...(-1)^{j-1}\tbinom{m+1}{j-1}(-1)^m](m-j+1) \\ &\quad + [\tbinom{m+1}{0}(j+1)^m - \tbinom{m+1}{1}(j^m) + \tbinom{m+1}{2}(j-1)^m \\ &\quad - ...(-1)^j\tbinom{m+1}{j-1}(1)^m](j+1) \end{aligned}$$

$$= [\tbinom{m+1}{0}(j^m)(m-j+1) - \tbinom{m+1}{1}(j-1)^m(m-j+1)$$
$$+ \tbinom{m+1}{2}(j-2)^m(m-j+1) - ...(-1)^{j-1}\tbinom{m+1}{j-1}(-1)^m(m-j+1)]$$
$$+ [\tbinom{m+1}{0}(j+1)^m(j+1) - \tbinom{m+1}{1}(j^m)(j+1) + \tbinom{m+1}{2}(j-1)^m(j+1)$$
$$- ...(-1)^j\tbinom{m+1}{j-1}(1)^m(j+1)]$$

We can pair the elements by their $(j-k)$ exponents such that they have the following form for $0 \leq k \leq j-1$:

$$(-1)^k\binom{m+1}{k}(j-k)^m(m-j+1) + (-1)^{k+1}\binom{m+1}{k+1}(j-k)^m(j+1)\ .$$

The only term that cannot be paired this way is the $\binom{m+1}{0}(j+1)^m(j+1)$ term from above. However, observe that $\binom{m+1}{0}(j+1)^m(j+1) = \binom{m+1}{0}(j+1)^{m+1}$. Since $\binom{r}{0} = 1$ for all natural numbers r, we have $\binom{m+1}{0}(j+1)^{m+1} = \binom{m+2}{0}(j+1)^{m+1}$, which is the first term of our desired result (when $k=0$). We will now manipulate the sum of each pair above to show we get the rest of our desired result. We notice that the two pieces that are paired together will always have opposite signs since the negative ones will have exponents of opposite parity. Thus,

$$\pm\tbinom{m+1}{k}(j-k)^m(m-j+1) \mp \tbinom{m+1}{k+1}(j-k)^m(j+1)$$
$$= (j-k)^m\left[\pm\tbinom{m+1}{k}(m-j+1) \mp \tbinom{m+1}{k+1}(j+1)\right]$$
$$= (j-k)^m\left[\pm\tbinom{m+1}{k}m \mp \tbinom{m+1}{k}j \pm \tbinom{m+1}{k} \mp \tbinom{m+1}{k+1}j \mp \tbinom{m+1}{k+1}\right]$$
$$= (j-k)^m\left(\left[\mp\tbinom{m+1}{k}j \mp \tbinom{m+1}{k+1}j\right] \pm \tbinom{m+1}{k}m \pm \tbinom{m+1}{k} \mp \tbinom{m+1}{k+1}\right)$$

Since $\binom{m+1}{k+1} = -\binom{m+1}{k} + \binom{m+2}{k+1}$ by Equation 4,

$$= (j-k)^m \left(\left[\mp\tbinom{m+1}{k} \mp \tbinom{m+1}{k+1}\right] j \pm \tbinom{m+1}{k}m \pm \tbinom{m+1}{k} \mp \left[-\tbinom{m+1}{k} + \tbinom{m+2}{k+1}\right]\right)$$

Since $\mp\binom{m+1}{k} \mp \binom{m+1}{k+1} = \mp\binom{m+2}{k+1}$ by Equation 4,

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}j \pm \tbinom{m+1}{k}m \pm \tbinom{m+1}{k} \pm \tbinom{m+1}{k} \mp \tbinom{m+2}{k+1}\right)$$

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}j \pm \tbinom{m+1}{k}(m+2) \mp \tbinom{m+2}{k+1}\right)$$

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}j \pm \tfrac{(m+1)!(m+2)}{k!(m+1-k)!} \mp \tbinom{m+2}{k+1}\right)$$

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}j \pm \tfrac{(m+2)!}{k!(m+1-k)!} \cdot \tfrac{(k+1)}{(k+1)} \mp \tbinom{m+2}{k+1}\right)$$

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}j \pm \tfrac{(m+2)!(k+1)}{(k+1)!(m+1-k)!} \mp \tbinom{m+2}{k+1}\right)$$

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}j \pm \tbinom{m+2}{k+1}(k+1) \mp \tbinom{m+2}{k+1}\right)$$

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}j \pm \tbinom{m+2}{k+1}k\right)$$

$$= (j-k)^m \left(\mp\tbinom{m+2}{k+1}(j-k)\right)$$

$$= \mp\tbinom{m+2}{k+1}(j-k)^{m+1}$$

We note that ultimately the sign of this term is $(-1)^{k+1}$ since it matches the sign of the second term in the original pairing. Thus we have

$$B_{m+1,j+1} = \binom{m+2}{0}(j+1)^{m+1} + \sum_{k=0}^{j-1}(-1)^{k+1}\binom{m+2}{k+1}(j-k)^{m+1} .$$

Shifting the index k by one in the summation gives us

$$B_{m+1,j+1} = \binom{m+2}{0}(j+1)^{m+1} + \sum_{k=1}^{j}(-1)^{k}\binom{m+2}{k}(j-k+1)^{m+1} .$$

Finally, combining these terms we get

$$B_{m+1,j+1} = \sum_{k=0}^{j}(-1)^{k}\binom{m+2}{k}(j-k+1)^{m+1} ,$$

which is our desired result. □

5 Further Explorations

The patterns explored in this paper were constrained to those that can be discerned from the first few rows of the Factorial Triangle and Euler's Number Triangle. However, as previously mentioned, these arrays have corresponding polynomials. Possible further studies could be done in comparing the patterns from the array to patterns seen in the corresponding polynomial in a "line-by-line" case. Another potential area of further study could include how do slight changes to the arrays affect the patterns. For example, how does changing the first entry to a number other than one affect the patterns? There is also room for further studies where the arrays contain negative entries and the patterns that may be seen then.

The final area of further study we can to discuss is patterns that may exist in later rows of Euler's Number Triangle and the Factorial Triangle. We showed that the patterns discussed here will work for any row (an value of i). However, it is possible that at greater values of i that new patterns are introduced that are not seen at lower values of i.

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