DISCOVERING SHADOWS IN HYPERBOLIC GEOMETRY

Lauren Vowinkel

Advisor: Dr. Andrew Lazowski

MA 398: Senior Seminar

Instructor: Jean Guillaume

Fall 2022

Abstract

Hyperbolic geometry is a type of non-Euclidean geometry that contradicts the fifth axiom of Euclid's Elements. We will discuss why this axiom was controversial and how hyperbolic space differs from Euclidean space in terms of shadows. The speed of how fast a person walks in terms of their shadow can be calculated using similar triangles or solved as a linear function in Euclidean space. For this type of problem, how fast the shadow is moving in Euclidean space yields a finite number. However, in hyperbolic space, the solution may not be finite.

1 Introduction to Euclidean Space

In this paper, we will discuss two types of geometry: Euclidean geometry and Hyperbolic geometry. We will consider the differences between them and by using a common related rates problem, we will explore how the distances obtained vary based on the geometry that is used to solve the problem. This paper is based on an article written by Ryan Hoban, called *Euclidean, Spherical, and Hyperbolic Shadows* [4]. In this article, Ryan Hoban explores three types of geometries when solving a common Calculus problem.

Geometry is defined as a type of mathematics that considers the relationship between all shapes, including lines, points, and surfaces. The specific geometry we use to describe how shapes are made up is called Euclidean geometry. Euclid was the first mathematician to organize and formalize it, around 300 B.C. In his book, *The Elements*, he provided definitions, postulates, and proofs that deal with geometry and other theories. He began with five postulates. These postulates, or axioms, are a variety of statements which are assumed to be true [1].

Here are the axioms:

1. The first postulate states "Let it have been postulated to draw a straight-line from any point to any point." This means that from any point on the plane, a, a straight line can be drawn to another point, b.

2. The second postulate states "And to produce a finite straight-line continuously in a straight line." This axiom describes that from any point on the plane, a, a line can be drawn-out indefinitely as a straight line.

3. The third postulate states "And to draw a circle with any center and radius," meaning that for any point, there exists another point not equal to the first, a circle can be drawn with a center at the original point and has a specific radius.

4. The fourth postulate states "And that all right-angles are equal to one another." This means that all right angles are congruent, or identical to each other.

5. The fifth axiom declares that for any line l and point a not on l, there exists exactly one line m that goes through a that is parallel to l. See Figure 1.

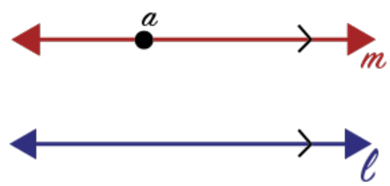

Figure 1: Parallel Postulate

The 5th axiom is called the parallel postulate and it became controversial. Previously, this was accepted as an axiom, meaning it needed no proof. However, people thought it wasn't an axiom, but a theorem that could be proved using other axioms. Many mathematicians failed to prove this statement so they began to wonder if the axiom could be changed. Eventually, it was discovered by Janos Bolyai and Nicholai Lobachevski that if you remove the axiom, one can create a new geometry, hyperbolic geometry.

Before we discuss hyperbolic geometry, we will think about Euclidean geometry using a metric, which is a way to measure distance, as opposed to building it with a set of axioms. This is a more modern way to describe geometry.

DEFINITION 1.1. *A metric d on a set X is a function*

$$d : X \times X \to \mathbb{R}$$

which also satisfies three conditions for any $x, y, z \in X$:

1. $d(x, y) \geq 0$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;
2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$; *and*
3. $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$ *(The Triangle Inequality).*

The metric that we are familiar with, the Euclidean metric, is defined using calculus. If c is a curve given by the equations $x = f(t)$, and $y = g(t)$, $a \leq t \leq b$, then

$$\text{length of } c = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

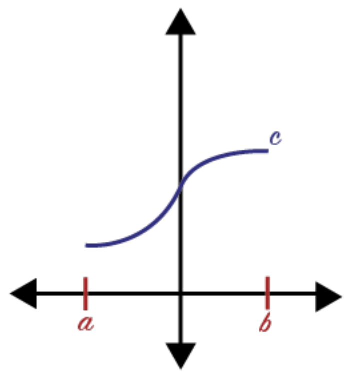

Figure 2: Curve from a to b

In order for any function to be a metric, it must satisfy the three conditions which are stated above. Here, we will look into some examples of metrics on $\mathbb{R}$.

EXAMPLE 1.2. *The function d(x,y) = $|x+y|$ where x,y $\in \mathbb{R}$ is a metric. This gives distance on the real line.*

1. $d(x,y) = |x+y| \geq 0$ by definition of absolute value. Now, suppose $d(x,y) = 0$, then $|x-y| = 0$, so $x = y$. The difference between two real numbers is zero only when they are the same. Suppose $x = y$, then $x - y = 0$, so $|x-y| = 0$ and $d(x,y) = 0$.

2. $d(x,y) = |x-y| = |(-1)(y-x)| = |-1||y-x| = |y-x| = d(y-x)$.

3. $d(x,y) + d(y,z) = |x-y| + |y-z| \geq |x-z| = d(x,z)$ by real analysis.

EXAMPLE 1.3. *The function d(x,y) = xy where x,y $\in \mathbb{R}$ is not a metric.*

1. Is $d(x,y) \geq 0$? Let $x = -1$ and $y = 2$, then $d(-1,2) = (-1)(2) = -2 < 0$. The function fails to be a metric because $-2 < 0$.

2 Related Rates Problem

Consider this related rates problem using similar triangles:

A street light is mounted at the top of a 15-ft tall pole. A man 6 feet tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole? ([2], pg 181) See Figure 3 for a diagram of the problem.

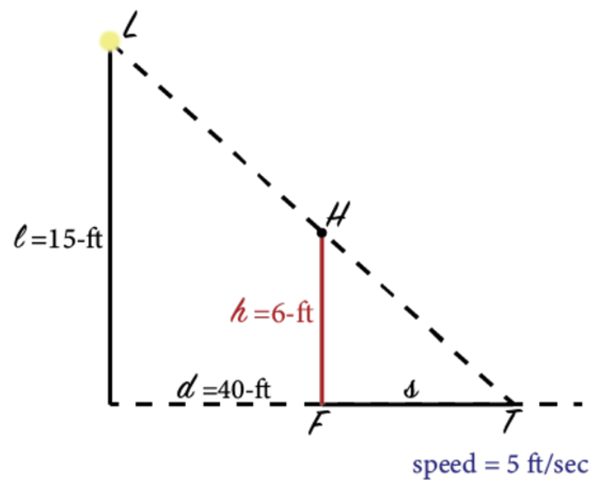

Figure 3: The Setup in the Euclidean Plane

Similar problems are frequently found in all calculus textbooks and would be solved using related rates. We can solve this by finding s in terms of d and F and differentiating to find the speed that the tip of the shadow moves.

Let the man have a fixed height, h, and the lamppost have a fixed height, l. We can also assume that the height of the lamppost is greater than the height of the man. Then, let s be the length of the shadow and d be the distance from the bast of the post to the man's feet. The example below performs all the calculations needed to solve the problem in the traditional way.

EXAMPLE 2.1. *Let* $l = 15$ *and* $H = 6$*, then* $s = F - d$

$$F - d = \left(\frac{6}{15}\right)$$

$$F = \frac{5}{3}d$$

$$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{5}{3}(5) = \frac{25}{3} = 8.33 ft/sec.$$

We can conclude that by walking a finite distance away from the lamppost of 40 ft, the tip of the shadow moves at a constant rate of 8.33 ft/sec.

The same problem can also be solved as a linear function, we do this in the next example.

EXAMPLE 2.2. *In order to solve the problem as a linear function, we still have to find the length of the shadow, s, as a function of the man's distance from the base of the lamppost, d. Assume that the ground is the x-axis and the lamppost is the y-axis, according to the common xy-coordinate plane.*

Using similar triangles, we can find the slope of line L to T to be:

$$\left(\frac{h-l}{d}\right).$$

Then, the equation of the line from L to T is:

$$y = \left(\frac{h-l}{d}\right)x + l.$$

Now, using the slope we can find the x-intercept, of the tip of the shadow, to be at the point

$$\left(\frac{ld}{l-h}, 0\right).$$

We can see this when we let $y = 0$

$$0 = \left(\frac{h-l}{d}\right)x + l$$

$$0 - l = \left(\frac{h-l}{d}\right)x$$

$$d(0-l) = (h-l)x$$

$$x = \frac{d(0-l)}{(h-l)} = \frac{-ld}{h-l} = \left(\frac{ld}{l-h}\right)$$

$$x = \frac{ld}{l-h}.$$

Next, we can find the length of the shadow to be the distance from the tip of the shadow to the man's feet:

$$s = dist(\text{feet, tip of shadow})$$

$$s = dist\left((d,0)\,, \left(\frac{ld}{l-h}, 0\right)\right)$$

$$\left(\frac{ld}{l-h}, 0\right) - (d,0)$$

$$\frac{ld}{l-h} - \frac{d(l-h)}{l-h} = \frac{dl}{l-h} - \frac{dl+dh}{l-h} = \frac{ld-dl+dh}{l-h} = \frac{dh}{l-h} = d\left(\frac{h}{l-h}\right) = s.$$

Now, we need to differentiate to find the rate of change of the length of the shadow and add that to the speed that the man is walking on the ground. We see that

$$s(d) = \frac{d}{dt}(d+s) = \frac{d}{dt}\left[d + \left(d\left(\frac{h}{l-h}\right)\right)\right] = \frac{dd}{dt} + \frac{d}{dt}\left(d\left(\frac{h}{l-h}\right)\right)$$

$$s = d\left(\frac{h}{l-h}\right)$$

$$5 + \frac{dd}{dt}\left(\frac{6}{9}\right) = 5 + \frac{30}{9} = \frac{25}{3} = 8.33ft/sec.$$

3 Introduction to Hyperbolic Space

In hyperbolic space, the fifth axiom in Euclid's *The Elements* is rejected.

DEFINITION 3.1. *The fifth axiom declares that for any line l and point a not on l, there exists exactly one line m that goes through a that is parallel to l.*

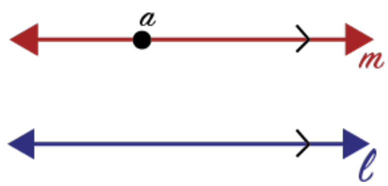

Figure 4: Recall the picture of the parallel postulate

3.1 The Upper Half Plane Model

Note that we live in Euclidean space and hyperbolic geometry is a non-Euclidean geometry [5]. This means that we must model hyperbolic space using Euclidean geometry. There are different models we can use and each has a different set of points to consider and a different definition of metric for that set. However, all of the metrics can be shown to be equivalent.

In hyperbolic space, more than one line can be parallel to another, contradicting the parallel postulate. We can see this when looking at an upper half plane model. Instead of a traditional $x - y$ coordinate plane, mathematicians use the upper half plane model in hyperbolic space. This plane is the upper half of the complex plane and is denoted using $\{x + iy : y > 0\}$. The set of points in hyperbolic space is defined as $\{(x, y) : y > 0\}$. Therefore, if a curve, c, is given by the equations $x = f(t)$, and $y = g(t)$, $a \leq t \leq b$, then

$$\text{length of } c = \int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y} dt.$$

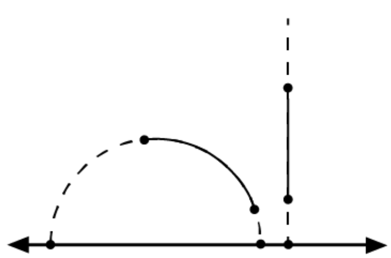

Figure 5: Upper Half Plane Model

Using this figure, we can draw some conclusions. The horizontal line in the upper half plane model is a real axis that we can approach, but will never be able to reach. Lines in the upper half plane model either extend to infinity or create what look like semi-circles, which shows that lines in the upper half plane will converge in one direction and diverge in another and these rays or arcs meet the real axis at right angles.

The bold lines in Figure 5 represent the shortest distance between the two points. This is either on the vertical ray or the circular arc. Specifically, on the circular arc, the shortest distance connecting the two points would not be a straight line between them, but instead would be the curve between them. We see this when we measure the distance between two points using a hyperbolic metric.

Using the figures below, we can look more into hyperbolic parallel lines [3].

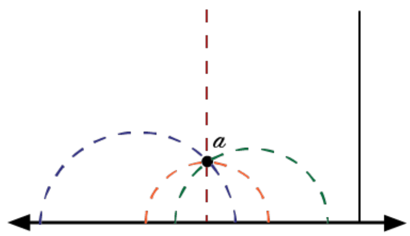

Figure 6: Hyperbolic Parallel Lines

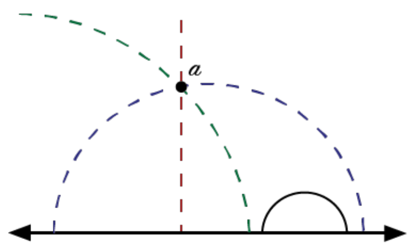

Figure 7: Hyperbolic Parallel Lines

There exists two types of hyperbolic lines. We see them as vertical rays or circular arcs in the above figures. In Figure 6, we can see that there belongs a point, a, not on the solid, black line and there exists more than one parallel line. Similarly, in Figure 7, we can see there belongs another point, a, not on the solid, black line line and there again, exists more than one parallel line.

In each figure, we can conclude that there is more than one line parallel to another in the upper half plane model.

3.2 Complex Notation

Through complex notation, the distance between two points in hyperbolic space can be found. A hyperbolic metric can be used to find the distance along a curve through complex notation.

DEFINITION 3.2. *Let $z, w \in \mathbb{H}^2$, then*

$$d(z,w) = \ln \frac{|z-\overline{w}| + |z-w|}{|z-\overline{w}| - |z-w|}.$$

EXAMPLE 3.3. *Let $z = 3+4i$ and $w = 1+i$, then*

$$d(z,w) = \ln \frac{|(3+4i)-(1-i)| + |(3+4i)-(1+i)|}{|(3+4i)-(1-i)| - |(3+4i)-(1+i)|} = \ln \frac{|(2+5i)| + |(2+3i)|}{|(2+5i)| - |(2+3i)|}$$

$$= \ln \frac{\sqrt{(4+25)} + \sqrt{(4+9)}}{\sqrt{(4+25)} - \sqrt{(4+9)}} = \ln \frac{\sqrt{29}+\sqrt{13}}{\sqrt{29}-\sqrt{13}}.$$

DEFINITION 3.4. *If these two points, z and w, are on the y-axis, then $z = ip$ and $w = iq$ and the formula is $d(ip, iq) = \left|\ln\left(\frac{p}{q}\right)\right|$.*

EXAMPLE 3.5. *Let $z = 5i$ and $w = 2i$, then $d(z,w) = \left|\ln\left(\frac{5}{2}\right)\right|$.*

3.3 Background Knowledge for the Problem in Hyperbolic Space

For the related rates problem, we will work in the upper half plane model. Using complex notation, we can find a formula for the distances. Given points in the upper half plane, z and w, $\mathrm{Re}(z) = \mathrm{Re}(w)$ when the hyperbolic line is a vertical line segment connecting z and w. When z and w lie on a circular arc, $\mathrm{Re}(z) \neq \mathrm{Re}(w)$. If any circular arc or vertical ray were to be extended, we can see that each will intersect $\mathbb{R} \cup \{\infty\}$ twice. The endpoints in each circular arc or vertical ray are called *ideal points*. These can be denoted with a * on their respective points. In this instance, they would be denoted z^* and w^*.

Now, let $\mathbb{HD}(z, w)$ be the hyperbolic distance between the points z and w, therefore

$$\mathbb{HD}(z,w) = \begin{cases} \ln\left(\frac{z-w^*)(w-z^*)}{(w^*-w)(z^*-z)}\right) & \text{if neither } z^* \text{ nor } w^* \text{ is } \infty, \\ \ln\left(\frac{Im(z)}{Im(w)}\right) & \text{if } z^* = \infty, \\ \ln\left(\frac{Im(w)}{Im(z)}\right) & \text{if } w^* = \infty. \end{cases}$$

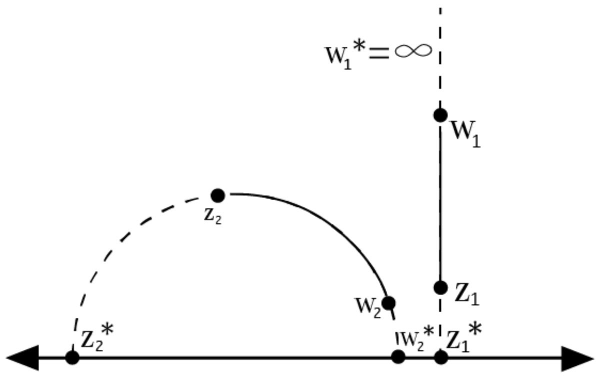

Figure 8: Typical Hyperbolic Lines in the Upper Half Plane

4 Related Rates Problem in Hyperbolic Space

Again, consider this related rates problem using similar triangles.

A street light is mounted at the top of a 15-ft tall pole. A man 6 feet tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

Just like we did when solving the problem as a linear function, we will find s as a function of d. Again, let the man have a fixed height, h, and the lamppost have a fixed height, l. We can also assume that the height of the lamppost is greater than the height of the man. Then, let s be the length of the shadow and d be the distance from the bast of the post to the man's feet. We can assume that the ground is the imaginary axis in the upper half plane model and the base of the lamppost lies at point i. We can assume that since the lamppost is perpendicular to the ground, then the curve of the lamppost lies on the unit circle. Looking at the above figure, we can see that the top of the lamppost L has a distance of l from i.

Using hyperbolic trigonometry, the point L can be written as $L = tanh(l) + isech(l)$.

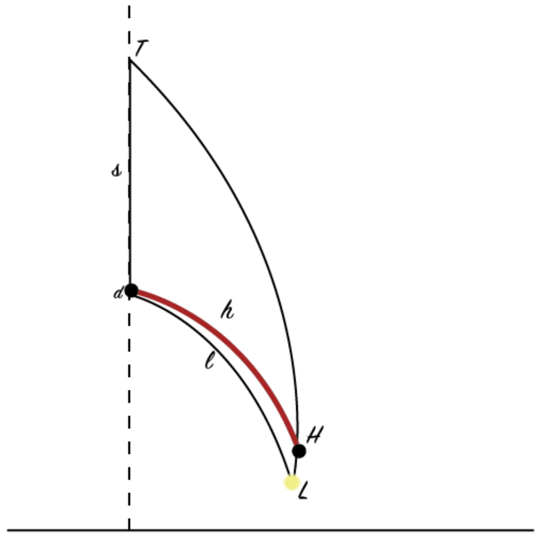

Figure 9: The Setup in the Upper Half Plane

Then, by looking at the figure, we can see that once that man starts to walk up, he walks a distance d and his feet would be at the point $F = ie^d$.

DEFINITION 4.1. *Hyperbolic sine and cosine satisfy: $cosh(x) + sinh(x) = e^x$. [6]*

Using Definition 4.1, we can find $F = ie^d$:

$$e^i\theta = cos\theta + isin\theta$$

$$e^i\theta = e^d + i(0)$$

$$e^i\theta = e^d + 0$$

$$e^i\theta = e^d + e^0 = e^d$$

$$F = ie^d.$$

Because we can assume that the man is standing straight up, that would mean his body lies along the circular arc putting his head at H and his feet at F. This arc would be centered at the origin with a radius of e^d and we can see this in the figure below.

Because the height of the man is h, his head lies on point H and H can be given by $H = e^d(tanh(h) + isech(h))$.

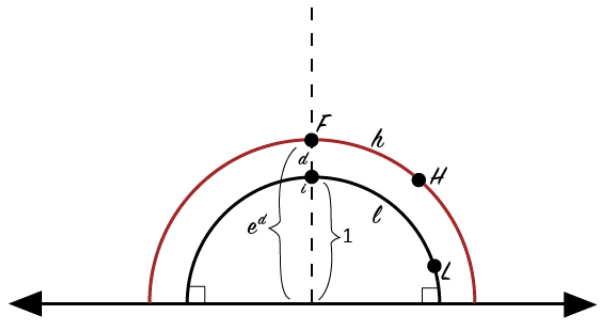

Figure 10: Another Setup in the Upper Half Plane

Using Figure 6, we can also see that the distance between i and F is d, the distance between i and L is l, and the distance between F and H is h.

Now, let T be the tip of the shadow. Then, the point T is the intersection of the ground with the ray of light from the lamppost.

By referring back to Figure 5, we can see that the ray of light is the arc through the points L and H.

We can call the point where that arc meets the real axis, x_0, based on previous knowledge that any arc in the upper half plane meets the real axis at right angles. Then, if r is the radius of that arc, then it satisfies the equation.

$$(x - x_0)^2 + y^2 = r^2.$$

Now, we can think of the equations of L and H in terms of $x + iy$. By replacing some parts of L and H in the equation, $(x - x_0)^2 + y^2 = r^2$, we get two new equations in terms of x_0 and r.

$$(tanh(l) - x_0)^2 + (sech(l))^2 = r^2$$

$$(e^d tanh(h) - x_0)^2 + (e^d sech(h))^2 = r^2.$$

With aid from a computer algebra system, we can find the equation of the tip of the shadow from those two equations above.

$$T = i\sqrt{\frac{e^d(tanh(h) - e^d tanh(l))}{e^d tanh(h) - tanh(l)}}$$

Now, we need to find the distance of s, or the distance from F to T.

$$s = \ln\left(\frac{i\sqrt{\frac{e^d(tanh(h) - e^d tanh(l))}{e^d tanh(h) - tanh(l)}}}{ie^d}\right)$$

$$s = \ln\left(e^{-d}\sqrt{\frac{e^d(tanh(h) - e^d tanh(l))}{e^d tanh(h) - tanh(l)}}\right).$$

Finally, we have s in terms of d. We can now see that when the numerator or denominator of s is zero, the equation is undefined. When the numerator or denominator is

$$d = d_0 = \pm ln\left(\frac{tanh(l)}{tanh(h)}\right)$$

the function is undefined.

We can see that:

$$s = ln\left(e^{-d}\sqrt{\frac{e^{\pm ln\left(\frac{tanh(l)}{tanh(h)}\right)}(tanh(h) - e^{\pm ln\left(\frac{tanh(l)}{tanh(h)}\right)}tanh(l))}{e^d tanh(h) - tanh(l)}}\right)$$

$$s = ln\left(e^{-d}\sqrt{\frac{0}{e^d tanh(h) - tanh(l)}}\right)$$

$$s = ln(0)$$

and s is undefined.

$$s = ln\left(e^{-d}\sqrt{\frac{e^d(tanh(h) - e^d tanh(l))}{e^{\pm ln\left(\frac{tanh(l)}{tanh(h)}\right)}tanh(h) - tanh(l)}}\right)$$

$$s = ln\left(e^{-d}\sqrt{\frac{e^d(tanh(h) - e^d tanh(l))}{0}}\right)$$

and s is undefined here as well.

Using the figures below, we can conclude that walking a fixed distance in either direction will give the man's shadow infinite length.

Here, when the man, or the red line, walks up, the man's height looks like it gets bigger, but in the hyperbolic plane, if the man kept walking, eventually, the tangent line to the man becomes a vertical ray. This shadow would turn into a vertical ray reaching toward infinity.

Similarly, when the man, or the red line, walks down, the man's height looks like it gets small, but in the hyperbolic plane, if the man kept walking, eventually, the tangent line to the man becomes a circular arc. This shadow would turn into a circular arc, which looks like it will intersect at the origin, but we know that when this happens, the arc doesn't ever intersect with that horizontal axis, but instead, extends toward infinity.

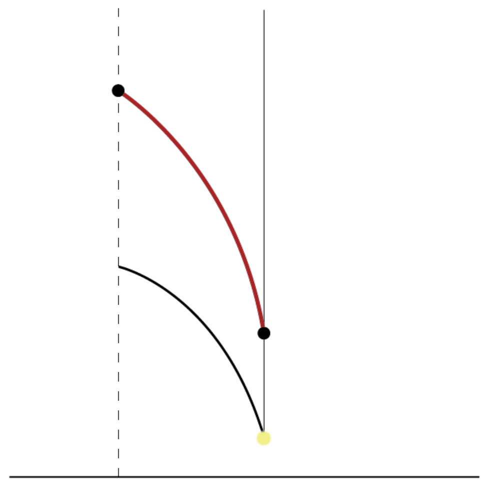

Figure 11: First Result

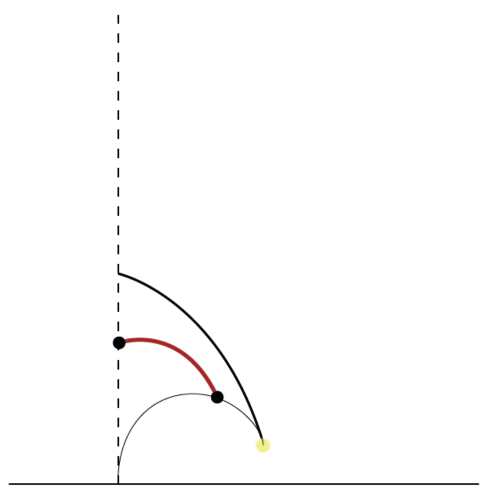

Figure 12: Second Result

5 Final Results

In conclusion, based off of the previous work, we can see that in Euclidean space, when the man walks away from the lamppost, his shadow becomes a finite distance, whereas in Euclidean space, when the man walks away from the lamppost, his shadow become infinite.

References

[1] A. Lazowski, "MA314: Geometry Lecture Notes." *Andrew Lazowski*, Sacred Heart University, Spring, 2022.

[2] J. Stewart, "Single Variable Calculus." *Thomson Brooks/Cole*, Belmont CA, 2005.

[3] R. Millman, "The Upper Half Plane Model for Hyperbolic Geometry." *The American Mathematical Monthly*, vol. 87, no. 1, 1980, pp. 48–53. JSTOR.

[4] R. Hoban, "Euclidean, Spherical, and Hyperbolic Shadows." *The College Mathematics Journal*, vol. 44, no. 2, 2013, pp. 126–34. JSTOR.

[5] Sami, "The Geometric Viewpoint, History of Hyperbolic Geometry, Colby College." *History of Hyperbolic Geometry*, 8 Dec. 2016.

[6] "Hyperbolic Functions." *Wikipedia*, Wikimedia Foundation, 1 Dec. 2022.