Tessellations in the World Around Us

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Abstract

Tessellations are a well-known concept seen all over the world. They are the notion of one or more geometric shapes being repeated several times on a plane with no gaps or overlaps. They are often seen in art, architecture, computer science, mapping, and more. There are also a few different varieties of tessellations, where some are more intricate than others. For example, the Voronoi tessellation is a special tessellation formed from a set of finite points on the Euclidean plane. Each of these finite points is called a seed which corresponds to a polygon, a Voronoi cell, which contains all points closer to that seed than any other seed in the tessellation. Voronoi tessellations can be applied and used in many aspects of daily life, including waste management, mapping, and various forms of science. This paper will explain the different forms of tessellations, including the Voronoi tessellation, their structure, the applications, and whether they would help to make the world a better place.

1 Introduction

Tessellations are a form of art and mathematics in which a tiling covers a surface with some form of plane shape such that there are no gaps or overlaps. Usually the tessellation involves a rotation, reflection, or translation of triangles, squares, or hexagons, but it can really involve any plane figure as long as there are no gaps or overlaps. Many times, tessellations are seen in computer science, but they are also seen in topology, as well as used to measure distances. Additionally, they are seen in architecture such as buildings, art, hobbies, mapping, and more. There are also special kinds of tessellations, like the Voronoi tessellation, where there exists several regions, or cells, formed from a set of finite points. Each of these points is considered a seed corresponding to one of the regions, or Voronoi cells, which contains all points closer to that seed than to any other seed in the tessellation. Voronoi tessellations can be seen in accordance with waste management as scientists have been trying to determine a more efficient way to manage waste around the world. Voronoi tessellations are also seen in relation to traveling, science, and other ideas.

This paper will start with a background on tessellations and where they are seen in real life. From there, we dive deeper into tessellations, specifically the Voronoi tessellation, and explore how they form, what their purpose is, and later on will explain how tessellations can be used in real-world situations like waste management, traveling, science, and more. In accordance with this, we explain how they can help to make the world a better place.

2 Tessellations

The word tessellation first came from the Latin word tessera, meaning small, tile like stone. Therefore, a tessellation is frequently seen in stone-work, mosaics, and other forms of art. Most commonly, a tessellation is known as a tiling which covers a surface using one or more plane shapes with no gaps or overlaps. We will now demonstrate which specific shapes can tessellate a plane.

THEOREM 2.1. *The only shapes that tessellate perfectly are a triangle, square, and hexagon.*

Proof. According to [6], first, consider a regular polygon such as a triangle, square, or hexagon. Recall that in order for a polygon to be regular, all of its sides must be of equal length, and each angle within the figure must be equivalent. Also recall that a regular polygon can be broken up into triangles, which are polygons consisting of 3 vertices with line segments connecting them together. Thus, a regular n-sided polygon can be broken into (n-2) triangles. Since all the internal angles of a triangle add up to 180°, the total of the internal angles for a regular polygon is $\frac{(180)(n-2)}{n}$. To determine which shapes tessellate perfectly, we need to determine how many times, T, the given shape tessellates. The product of these two must equal 360°such that

$$\frac{(180)(n-2)}{n} \times T = 360$$

By distributing and simplifying, we have

$$nT - 2T = 2n \tag{2.1}$$

By adding 4 to each side of the equation, we obtain the following equation:

$$nT - 2T + 4 = 2n + 4 \tag{2.2}$$

From there, we want to subtract $2n$ from the right and rearrange the variables such that we obtain

$$nT - 2n - 2T + 4 = 4$$

. Now, we can factor these segments out so that

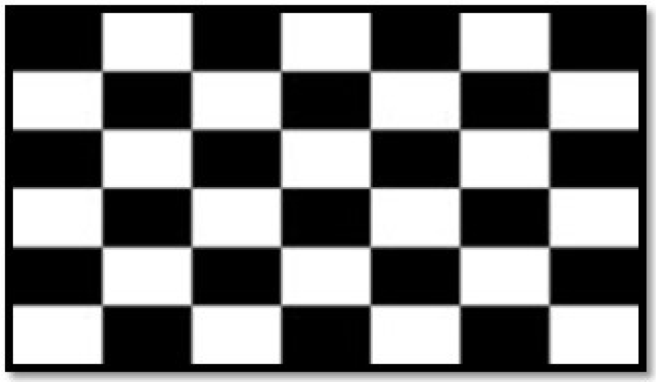
$$(n-2)(T-2) = 4 \tag{2.3}$$

From this, the only possible solutions are $\{3, 6\}$, $\{4, 4\}$, or $\{6, 3\}$, which correspond to a triangle, square, or hexagon. Therefore, these are the only regular polygons that perfectly tessellate. □

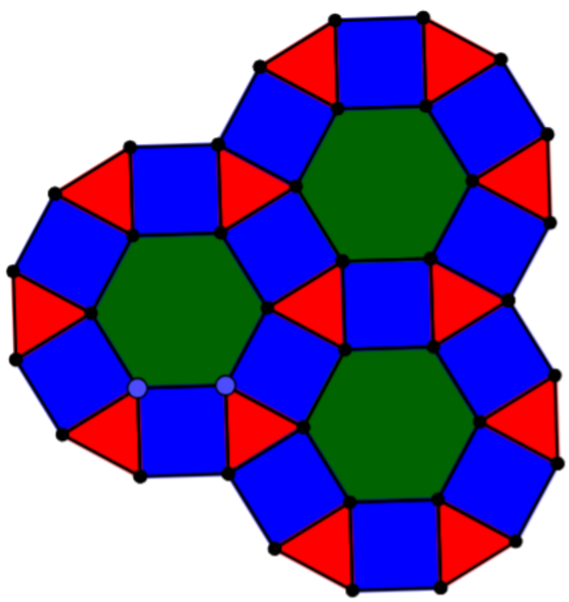
2.1 Different Types

Tessellations come in various forms, some more complex than others. The two most common forms of tessellations are regular and semi-regular tessellations.

In a regular polygon tessellation, there is a series of repeating polygon shapes tiling such that each polygon meets vertex to vertex [5]. This means typically only one plane shape is used for the tessellation. One very common example of this type of tessellation would be a checkerboard, as seen below, where each polygon meets vertex to vertex.



In a semi-regular polygon tessellation, two or more regular polygons are arranged such that each vertex point is identical. In other words, every vertex is arranged with the same polygons in the same order. Some examples of semi-regular tessellations can be seen below.



3 Applications

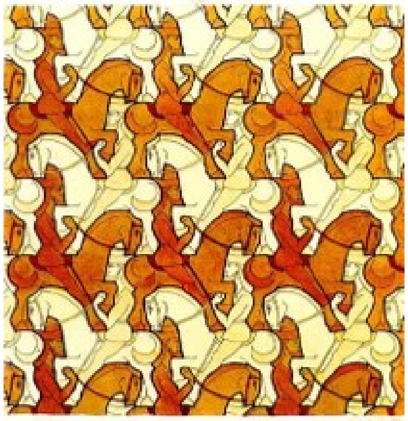
3.1 History

Tessellations is a major idea that goes back thousands and thousands of years. Some historians have recorded tessellations being used to build wall decorations and clay tilings as far back as 4000 B.C. There is also research suggesting that a tessellation technique was used by Sumerians in 5th and 6th BC to decorate homes, but others such as Egyptians, Persians, Romans, Greeks, Arabs, Japanese and more have adopted the idea since [5].

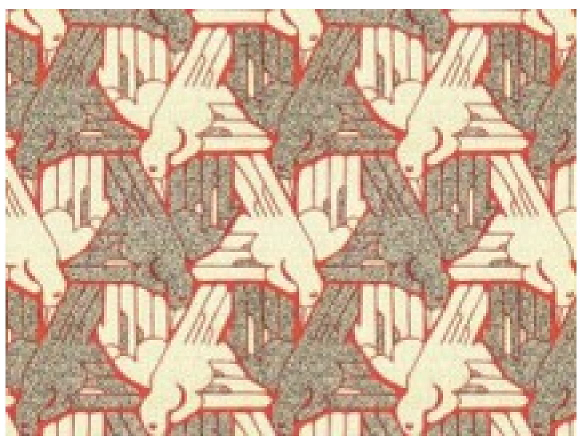
Transitioning thousands of years, in 1619, mathematicians and scientists began studying honeycombs and snowflakes, specifically looking at their structure. This would be one of the first beginnings to the studying of tessellations. This studying would continue to develop until 1891, when the Russian crystallographer Yevgraf Fyodorov was able to prove that every "periodic tiling of the plane features one of seventeen different groups of isometries."[5] This would be the unofficial beginning of the study of tessellations. Finally, in the early 20th century, tessellations were named for the Ukrainian mathematician, Georgy Feodosevich Voronoi.

3.2 Art

Tessellations are seen in various forms of art. One well-known artist by the name of M.C. Escher, is known as the father of modern tessellations. Despite him having almost no background in mathematics, he created various pieces typically incorporating animals and fun designs and had an eye for geometry. His main goal was to get balance, harmony, and perfection in his works. He used to say that "order is repetition of units. Chaos is multiplicity without rhythm."[5]

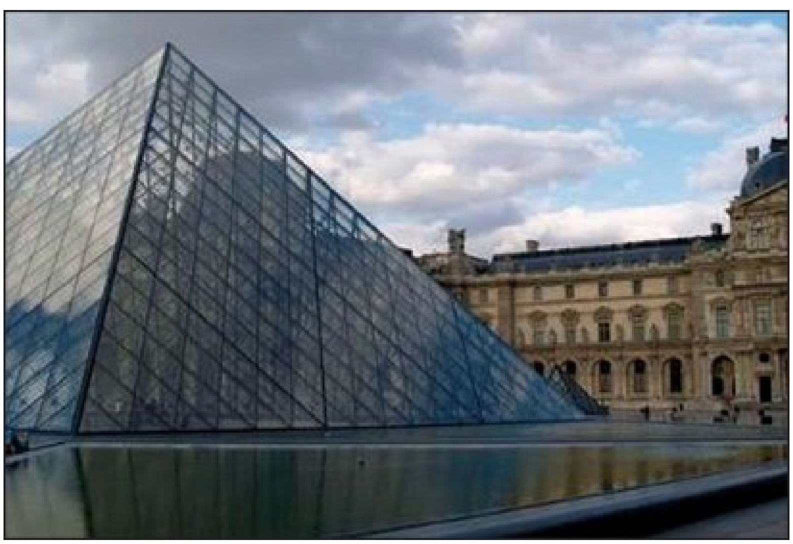


Another well-known artist by the name of Koloman Moser was an Austrian artist and used the idea of tessellations with several of his works. Many of his works contain art nouveau style patterns. He also has many works that include nature motifs and plane shapes, similar to those of Escher. Unlike Escher, however, many of Moser's works are used for textiles in fashion, interiors, walls, posters and more.



Not only are tessellations used by many artists, but they are also seen in Islamic art and architecture. This style is most commonly known for their absence of figures and other beings. Instead of using these images, they use geometric tiles. One of the most famous architecture pieces in all of Islamic culture is Alhambra, a huge palace located in Granada, Spain, shown below. Many of their various styles of tile work is seen all over the world. Much of their tile work is based on either interlacing patterns, calligraphy, or a combination of the two.[1] The interlacing patterns incorporate more shapes or arabesques while calligraphy incorporates more elements of nature such as trees, leaves, branches, and more. One of their more well-known techniques when it comes to tiling is with Zellij, or Zillij or Zellige, as seen below. This style is most often found in Morocco and other Islamic countries on walls and floors of multiple kinds. This is a mosaic type work made up of small, hand-chiseled elements put into a plaster base. It is a more intricate design, typically involving several shapes, with a complicated tiling process. In the end however, it becomes a wonderful display.


Tessellations can also be seen in many forms of modern art. Several art museums, or even graffiti on buildings, or mosaics would all be modern forms of tessellations. Additionally, any form of brick or stone building is a tessellation. Some famous examples of this would be the Louvre in Paris, France. The outside structure of the building is made of diamonds, which is built off of triangles, which contains no gaps or overlaps. This is just one example, but there are hundreds of others around the world, including a few galleries and buildings in England.



3.3 Topology

Tessellations are also seen and used in fields such as topology. Topology is the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures. Topology also includes one mathematical element known as an isometry.

An isometry of $\mathbb{R}$ is a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which preserves Euclidean distance, that is, if d represents the Euclidean metric in $\mathbb{R}$ then $d(f(a), f(b)) = d(a, b)$. In other words, the location of a shape can change as long as it maintains its same size and shape. There are many ways in which isometries are viewed. For instance, isometries are built off of reflections, translations, or rotations, all of which are built off of reflections. Furthermore, a typical reflection is made up of one reflection, or a more special form of reflection, the glide reflection, is made of three reflections, a translation is made up of two reflections, and a rotation is made up of two reflections. Therefore, whenever a triangle or another polygon is rotated, translated or reflected to form a tessellation, that is a form of isometry. Although they are not seen so much in this particular example, we can classify isometries based on how many reflections are involved. For instance, if an isometry is made up of an even number of reflections, it is known as an *orientation preserving* isometry, but if an isometry is made up of an odd number of reflections, it is known as an *orientation reserving* isometry.

4 Special Kind of Tessellation

4.1 Voronoi Tessellation

DEFINITION 4.1. *We are given a set of points, p in the Euclidean plane. Each site p in this case is a point, and its corresponding Voronoi cell, S contains every point in the Euclidean plane whose distance to s is less than or equal to its distance to any other point s′. Furthermore,*

$$Vor(s) = p : d(s, p) \leq d(s\prime\ , p), \forall s\prime \in S \tag{4.1}$$

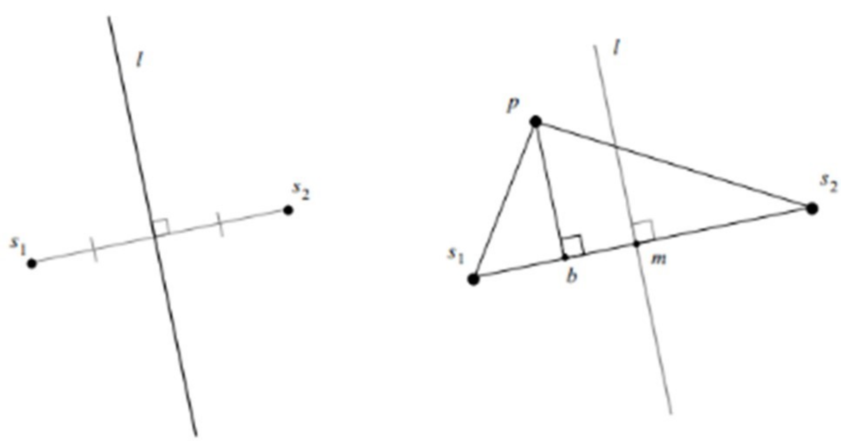
Before we think about the tessellation as a whole, we need to take a step back and look at tessellations from their starting point, and see how they would evolve over time.

4.1.1 How to Create a Voronoi Diagram

Start with a singular point, s_1, and the point's Voronoi region would be the entire plane it is in, S. Next, add a second point to the set, s_2, such that $S = \{s_1, s_2\}$, as seen in Figure 4.2a, where the Voronoi region consists of two half-planes that are divided by a ray l that is a perpendicular bisector for s_1 and s_2. One important aspect to note is the idea that these regions are not disjoint, but rather overlap at the points equidistant from the point and the ray, as shown in Figure 4.2a.

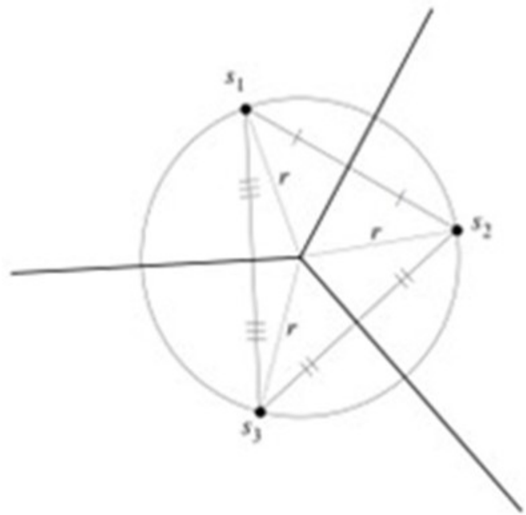
THEOREM 4.2. *According to [4], all points on the half plane containing s_1 and delimited by the perpendicular bisector l of $s_1 s_2$ are closer to s_1 than s_2.*

Proof. Consider a point p in the half-plane containing s_1 (Figure 4.2b). We can construct two right triangles $\triangle s_1 pb$ and $\triangle s_2 pb$ (Figure 4.3b). They share side $\overline{pb}$, and we see that $\overline{s_1 b}$ is shorter than $\overline{bs_2}$ since $||s_1 m||=$ $||s_1 b|| + ||bm||$ and $||s_1 m|| = ||s_2 m||$. The hypotenuse of $\triangle s_1 pb$, $\overline{s_1 p}$ is shorter than $\overline{s_2 p}$ by the Pythagorean theorem. Therefore p is closer to s_1 than s_2. □



(a) (b)

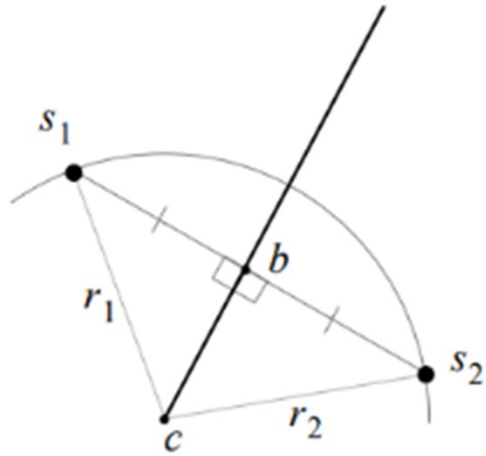
We now know that two points can help to start to form the area we need to create the tessellation. Next, we will show how to incorporate a third point, s_3, into the tessellation, using the image below (Figure 4.3).



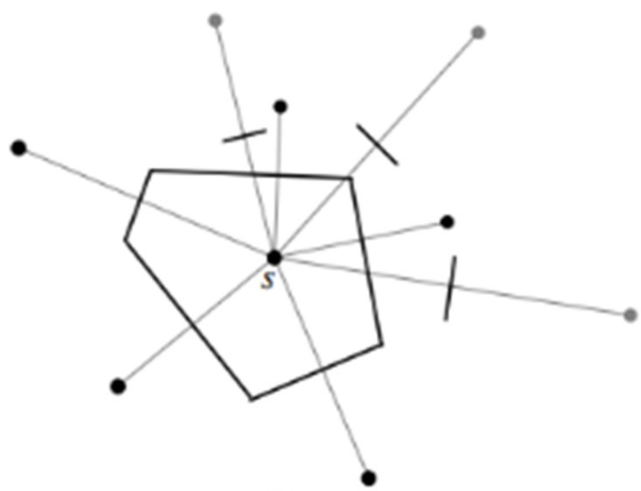
Since we have 3 points, we start by creating intersections between each pair of points. For example, connecting s_1 to s_2 or s_2 to s_3. This will form a triangle. From there, we add the perpendicular bisectors for each line, which intersect at one center point. After drawing the lines, erase the portion of each line going from the vertices, s_1 to s_2 or s_3, to the intersection point. At this point, the figure is complete. Notice that this point where the rays intersect is in the Voronoi region for each point. Next, [4] will prove that the meeting point for these three points, s_1, s_2, and s_3 is the center of the circle, c.

THEOREM 4.3. *The intersection of the 3 perpendicular bisectors for the segments* $\overline{s_1s_2}, \overline{s_2s_3}$, *and* $\overline{s_1s_3}$ *is the center of the circle containing the points* s_1, s_2, *and* s_3.

Proof. Any point on the perpendicular bisector, $\overline{pb}$, is equidistant from the two points it bisects. Therefore, segments $\overline{s_1b}$ and $\overline{s_2b}$ are equal (Figure 4.4). Angles $\angle s_1bc$ and $\angle s_2bc$ are right angles and both triangles share the side $\overline{bc}$. Therefore the two triangles, $\triangle s_1bc$ and $\triangle s_2bc$, must be congruent and their hypotenuses r_1 and r_2, which represent the radius of the circle, are equal. A similar argument can be made between s_3 and either of s_1 or s_2. Therefore, point c is equidistant from s_1, s_2, and s_3 and is thus the center of the circle containing s_1, s_2, and s_3. [4] □



We now see how this region, or triangle, has a center point c. Next, we want to form the Voronoi region for s. To do so, we first want to create line segments connecting s to the remaining members of S. Next, in order to form the Voronoi region for s, we want to take the perpendicular bisectors of the segments connecting s to the remaining members of S, then use these rays to determine the half-planes for each corresponding point. In other words, we will add in the perpendicular bisectors for each of the segments connecting s to the other members of S. At this point, if we extend each perpendicular bisector, we see that they intersect one another. This will form the border for the Voronoi region and hence, the intersection of all the half planes containing s is the Voronoi region for s, as seen in figure 4.5 below.

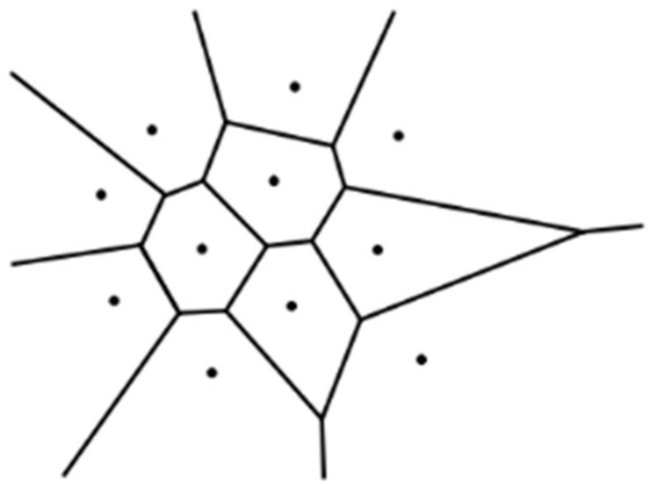


(a)

When finished, we see that a complete Voronoi diagram (figure 4.6) is when there is a union of all the

Voronoi cells, $Vor(s)$ in the set such that

$$Vor(s) = \bigcup_{s \in S} Vor(s)$$



This is just one partition containing only 11 seed points, but Voronoi tessellations can extend to much larger depths.

5 Special Applications

We have now seen how Voronoi tessellations form and how intricate they truly are. Next, we want to look at when they are seen in the real world. We already know that they are seen in art such as mosaics and architecture, in history, and in topology. There are, however, many more abstract ways in which tessellations are used every day in the world around us.

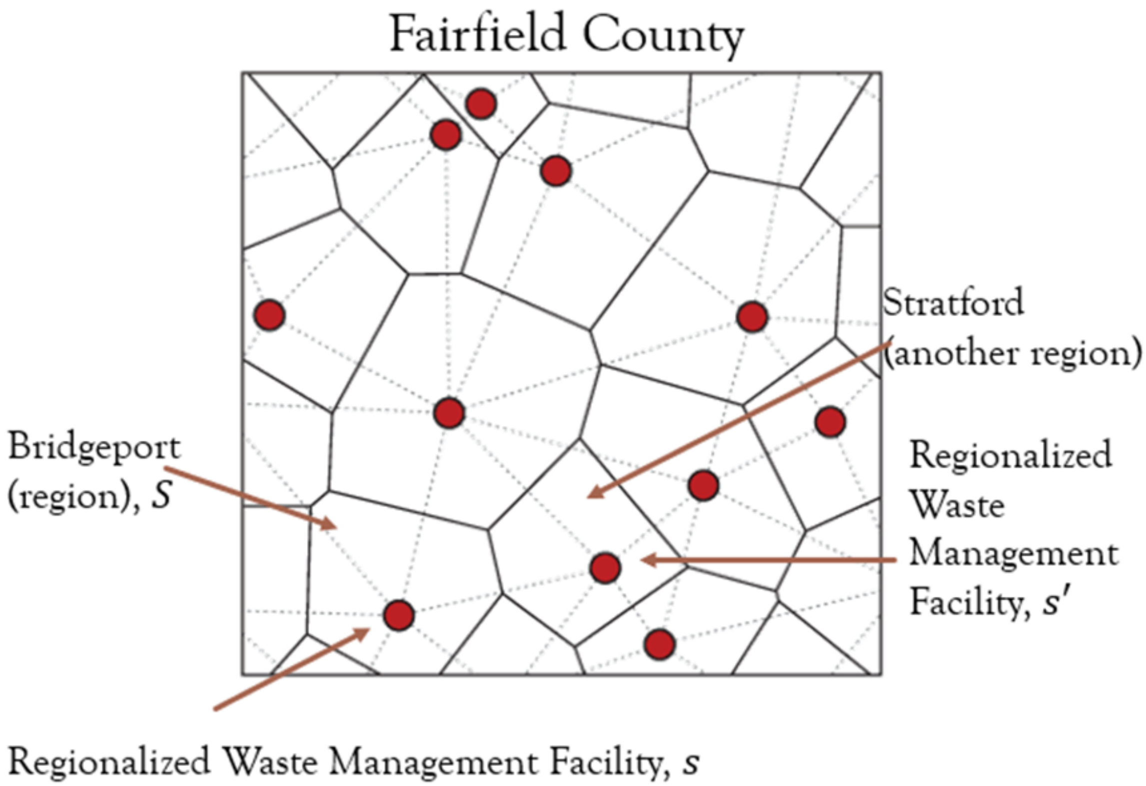
5.1 Waste Management

The process of the disposal of waste has been in practice since as early as 500 BC. Over time, however, between Industrialization, worsening health conditions with the Cholera outbreak, war, and increasing amounts of chemicals and toxicity in the atmosphere, the waste management issue worsened dramatically. As a result, our planet became very unhealthy and with that, thousands wanted change. Today, especially in poorer areas of the world, people might not have access to dump their trash in a specific area, so trash is thrown where it should not be, like on the streets, in overflowing landfills, the oceans, populated areas, and more. Additionally, the costs to manage waste are increasing constantly, so the problem is just worsening day by day. Thankfully, efforts across the globe are increasing.

5.1.1 How Does this Connect to Tessellations

First, picture a map. On maps, we see countries, states, counties, and towns or villages. Typically, when broken down, each town is a "cell" and there could be many "cells" in a county and even more in a given state. The idea behind the Voronoi tessellation, in this case, is scientists want to pinpoint a particular spot in the "cell" or region where people can go deposit trash. This could be a landfill, a garbage depot, or some other designated area. In her 2019 article, "Optimization of waste management regions using recursive Thiessen polygons", Amy Richter proposes creating one or more Regionalized Waste Management Facilities (WMF) in order to minimize the number of dumps [3]. In this process, the regions would account for any landfills, or permanent trash dumping areas, populated places, or neighborhood locations where trash is dumped, and road lengths. There could also be many of them in a given region if the area is large enough and more are needed. Even so, one would want to find the closest WMF to them and should not have to travel miles and miles to dispose of their trash. The main idea and goal of this plan is to eliminate waste from unwanted areas of the planet. Hopefully, if more areas of the world adopt this idea, we could possibly have a cleaner planet overall.

For example, we can consider the larger region of Fairfield County in Connecticut, which contains many smaller regions, or towns, within. Let us consider the city of Bridgeport in the southern part of the county S, and say this region contains a WMF, s. Now, let us consider the next town over, Stratford, $S\prime$, and its WMF, $s\prime$. If someone lived in Bridgeport, they would want to go to the WMF in Bridgeport to deposit their waste since it is the closest one to them. Unless they had to go to Stratford for some other reason, they would not want to go that far since the WMF in Bridgeport, s, is closer to them than the WMF in Stratford, $s\prime$.



5.2 Airport Locations

Another idea that incorporates Voronoi tessellations is airlines. Say that you are a pilot of a massive aircraft carrying hundreds of passengers, but there is a mechanical issue mid flight and you are forced to make an emergency landing. In this instance, the pilot would have to quickly make a decision as to which airport to land at. This is where Voronoi tessellations come in.

5.2.1 Tessellations Connection

When flying an aircraft, a pilot uses a map to detect the closest airport based on their location. A map can be split into regions, or cells, with an airport in each region. Based on where the plane is in the air, the pilot can use the tessellations to determine the closest airport to them so they can make the quickest and safest landing. This map could help to visualize the closest airports so the planes can land with their passengers more quickly and as safely as possible.

5.3 Other Applications of Voronoi Tessellations in Mapping

There are a few additional ways Voronoi tessellations are seen with mapping. For example, they are used in London with the tube stations to help travelers, conductors, and other individuals to find the closest tube stations to them. Also, these tessellations were used in relation to the Cholera outbreak. They were first used to detect how many deaths there were in a given area, and which pumps of drinking water they got the disease from. From there, John Snow, a medical physician who help develop anaesthesia and medical hygiene, used the map to find the source of the most disease and as a result, ended the Cholera outbreak. Tessellations have also been seen in Anthropology, Archaeology, and Astronomy. In anthropology, it was used to examine the similarities and differences in regard to plant and animal competitions. In archaeology, it is used to detect areas that are under the influence of certain clans. It can also be used in Geography and Marketing to map areas based on sparse samples. Finally, one of the more interesting applications was used by Descartes in 1744 to describe the influences of regions or galaxies and stars.

As you can see, Voronoi tessellations have a variety of applications to them, and have helped to change the world for the better in several ways. There may also be applications to the study that have not yet been discovered.

6 Conclusion

Tessellations are seen in many variations and are very versatile. They can be seen in several forms of designs including architecture, art, and many other aspects of life. They can also be used to solve world problems and help keep the citizens of the world safe like managing our planet's waste, detecting air travel, combating world-wide, horrible disease, and more. We were also able to see how intricate and fine the structure of the Voronoi tessellation truly is, particularly how important the connections between each site are and their corresponding perpendicular bisectors in forming each Voronoi region. Despite how particular each Voronoi tessellation is, they can either be very small or huge in depth. In any case, whether it is clearly visible or not, tessellations are hugely important in the world around us.

Since tessellations can be used for travelling, sciences, and waste management, with some research, it is possible there are more applications of it related to those aspects. Additionally, since tessellations are found so often in computer science, it is possible that they could help to develop more advanced forms of technology to help better our world. Furthermore, if another study is done about tessellations, it might be worth delving deeper into different applications, or further examining why these modes of application, like waste management and airline difficulty, have not been further implemented. Another aspect that might be worth looking into is whether proximity to another airport has anything to do with the number of plane crashes every year. In any case, we can see how important tessellations are in daily life and hopefully one can find more applications so that they can be used more often to improve daily life.

References

[1] "Islamic Tile History and Inspiration." Why Tile, 14 Dec. 2021, https://whytile.com/tile-history/islamic-tile-history-and-inspiration/.

[2] Keeler, Paul. "Voronoi Tessellations." H. Paul Keeler, 29 Dec. 2021, https://hpaulkeeler.com/voronoi-dirichlet-tessellations/.

[3] Richter, Amy, et al. "Optimization of Waste Management Regions Using Recursive Thiessen Polygons." Journal of Cleaner Production, vol. 234, Oct. 2019, pp. 85–96. EBSCOhost, https://doi-org.sacredheart.idm.oclc.org/10.1016/j.jclepro.2019.06.178.

[4] Snibbe, Scott S. Introduction to Voronoi Diagrams - Brown University. 22 Mar. 1993, https://cs.brown.edu/courses/cs252/misc/resources/lectures/pdf/notes09.pdf.

[5] Taggart, Emma. "Take a Tour of Tessellations, the Mathematical Art of Repeating Patterns." My Modern Met, 20 Sept. 2021, https://mymodernmet.com/tessellation-art/.

[6] "Tessellating Regular Polygons." Tessellating Regular Polygons, https://datagenetics.com/blog/september22019/.

[7] "Voronoi Tessellations." Voronoi Tessellations, Data Genetics, May 2017, https://datagenetics.com/blog/may12017/index.html.