The Dynamics of the Logistic Map and Difference Equations

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Abstract

Difference equations describe the evolution of a quantity or population whose changes are measured over discrete time intervals. In this presentation, we will investigate these types of recursive relations and classify the local stability of their equilibrium points and periodic solutions. In particular, we will examine the dynamics of the logistic map and the long-term behavior that occurs when we modify its parameter.

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Note that it is NOT the n-th derivative of $f(x_0)$.

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Fibonacci Recurrence: $x_{n+1} = x_n + x_{n-1}$, where $x_{-1} = 1$, $x_0 = 1$

Solution: $\{1, 1, 2, 3, 5, ...\}$, which forms the Fibonacci Sequence

Equilibrium Points

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If we let $x_0 = \overline{x}$, then we will stay at that same point for all iterations of the difference equation.

Stability Analysis

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If $|f'(\overline{x})| > 1$, then the equilibrium point is unstable.

*If $|f'(\overline{x})| = 1$, then we are working with non-hyperbolic difference equations, which expands into the study of *semistability analysis**.

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If we wanted to find the period-2 solutions, we would have to calculate $f^2(\overline{x}) = \overline{x}$ and solve for $\overline{x}$.

In general, if we wanted to find the period-n points, we would compute $f^n(\overline{x}) = \overline{x}$ and solve for $\overline{x}$.

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If we compile the minimal period-n points into a set, we have a *minimal period-n orbit*.

Tent Map

The *tent map* difference equation, defined by $x_{n+1} = T(x_n)$, is given by the function:

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

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Equilibrium points (minimal period-1 solutions):
$\overline{x} = 2\overline{x} \iff \overline{x} = 0$
$\overline{x} = 2 - 2\overline{x} \iff \overline{x} = \frac{2}{3}$

Tent Map

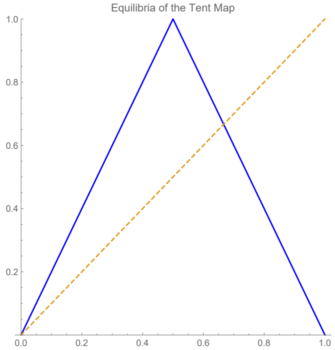

Figure: Graph of the tent map $T(x)$ and its equilibrium points. Minimal period-1 orbit: $\{0, \frac{2}{3}\}$.

Tent Map

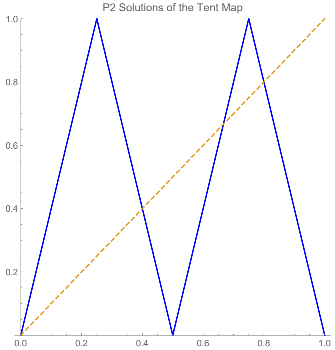

Figure: Graph of the second iteration of the tent map $T^2(x)$ and its period-2 solutions. Minimal period-2 orbit: $\{\frac{2}{5}, \frac{4}{5}\}$.

Tent Map

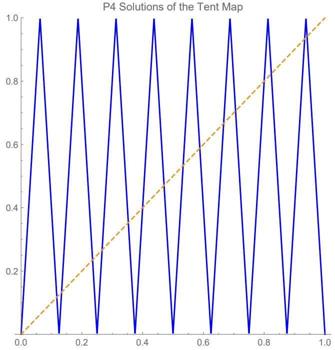

Figure: Graph of the fourth iteration of the tent map $T^4(x)$ and its period-4 solutions. Minimal period-4 orbits: $\left\{\frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{16}{17}\right\}$, $\left\{\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{14}{15}\right\}$, and $\left\{\frac{6}{17}, \frac{12}{17}, \frac{10}{17}, \frac{14}{17}\right\}$.

Basin of Attraction

Basin of attraction: points that converge to an equilibrium point as we perform iterations of the difference equation.

$$\mathcal{B}(\overline{x}) = \left\{ x \in D : \lim_{n \to \infty} f^n(x) = \overline{x} \right\}$$

where D is the domain of the function $f(x)$.

Logistic Map

The logistic map is given by the equation

$$x_{n+1} = \mu x_n(1 - x_n)$$

where $\mu > 0$, $x_0 \geq 0$. We will examine the domain $D = [0, 1]$, meaning $0 \leq x_0 \leq 1$.

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Equilibrium points:

$\overline{x} = \mu\overline{x}(1 - \overline{x})$

Solving for $\overline{x}$, we have $\overline{x} = 0$ and $\overline{x} = \frac{\mu-1}{\mu}$, where $\mu > 1$

Logistic Map

If $0 < \mu < 1$, then the sequence converges to the zero equilibrium.

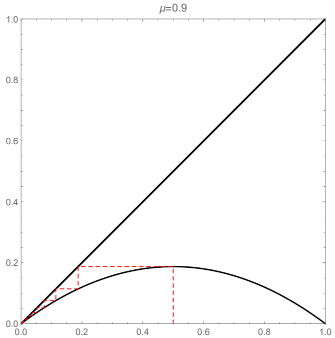

Figure: Graph of the logistic map with $0 < \mu < 1$. In this case, we have $x_{n+1} = 0.9x_n(1 - x_n)$, where $\mu = 0.9$ and $x_0 = 0.5$.

Logistic Map

If $1 < \mu < 3$, then the sequence converges to the positive equilibrium $\overline{x} = \frac{\mu - 1}{\mu}$.

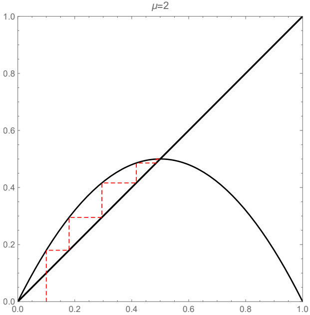

Figure: Graph of the logistic map with $1 < \mu < 3$. In this case, we have $x_{n+1} = 2x_n(1 - x_n)$, where $\mu = 2$, $x_0 = 0.1$, and the positive equilibrium $\overline{x} = \frac{1}{2}$.

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Say $z \in \left(0, \frac{\mu-1}{2\mu}\right)$: Logistic map increases over this interval and $|f'(x)| > 1$. By the Mean Value Theorem,
$f^r(z) = f(f^{r-1}(z)) < f\left(\frac{\mu-1}{2\mu}\right) = \frac{\mu^2-1}{4\mu} \leq \overline{x}$.

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$f^r(z) = f(f^{r-1}(z)) < f\left(\frac{\mu-1}{2\mu}\right) = \frac{\mu^2-1}{4\mu} \leq \overline{x}$.
Hence, $z \in \left(0, \frac{\mu-1}{2\mu}\right) \subset \mathcal{B}(\overline{x})$.

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Thus, $0 = f(1) < f(p) < f\left(\frac{\mu+1}{2\mu}\right) = f\left(\frac{\mu-1}{2\mu}\right) = \frac{\mu^2-1}{4\mu} \leq \overline{x}$.

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Thus, $f\left(\frac{\mu+1}{2\mu}\right) \subset (0, \overline{x})$, so since the logistic map is decreasing across this interval, $p \in \left(\frac{\mu+1}{2\mu}, 1\right) \subset \mathcal{B}(\overline{x})$.

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Thus, $f\left(\frac{\mu+1}{2\mu}\right) \subset (0, \overline{x})$, so since the logistic map is decreasing across this interval, $p \in \left(\frac{\mu+1}{2\mu}, 1\right) \subset \mathcal{B}(\overline{x})$.

Therefore, since $\left(0, \frac{\mu-1}{2\mu}\right) \subset \mathcal{B}(\overline{x})$, $\left[\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right] \subset \mathcal{B}(\overline{x})$, and $\left(\frac{\mu+1}{2\mu}, 1\right) \subset \mathcal{B}(\overline{x})$, it follows that $\mathcal{B}(\overline{x}) = (0, 1)$.

Logistic Map

If $3 < \mu < 1 + \sqrt{6}$, then the sequence converges to the period-2 solutions.

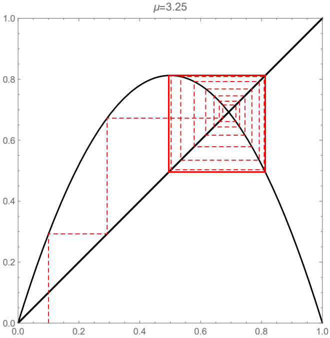

Figure: Graph of the logistic map with $3 < \mu < 1 + \sqrt{6}$. In this case, we have $x_{n+1} = 3.25x_n(1 - x_n)$, where $\mu = 3.25$ and $x_0 = 0.1$.

Bifurcation Values

When μ is equal to 1, 3, and $1+\sqrt{6}$, we notice a change in the long-term behavior of the logistic map.

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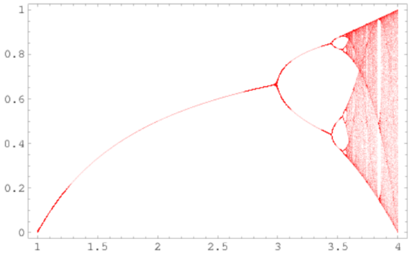
Thus, we call these quantities *bifurcation values*, since those points exhibit a shift in the qualitative dynamics of the logistic map.

Bifurcation Values

The bifurcation values of the logistic map can be represented in a bifurcation diagram.



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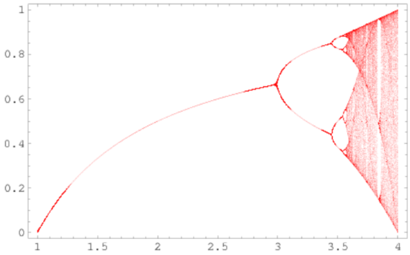

Period-doubling bifurcation route to chaos: If μ exceeds the k-th bifurcation value, the minimal period-$2k$ solutions are the only stable attractors.

Feigenbaum's Constant

If we put the bifurcation values into a set $\{\mu_n\}$, then we can find *Feigenbaum's Constant* by calculating

$$\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \approx 4.6692$$

n	μ_n	$\mu_n - \mu_{n-1}$	$\frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$
0	3	—	—
1	3.449499 ...	0.449499 ...	—
2	3.544090 ...	0.094591 ...	4.752027 ...
3	3.564407 ...	0.020313 ...	4.656673 ...
4	3.568759 ...	0.004352 ...	4.667509 ...
5	3.569692 ...	0.00093219 ...	4.668576 ...
6	3.569891 ...	0.00019964 ...	4.669354 ...

Chaos Theory

When $\mu > 3.56994$, the logistic map will experience *chaos* and exhibit inconsistent patterns within the sequence.

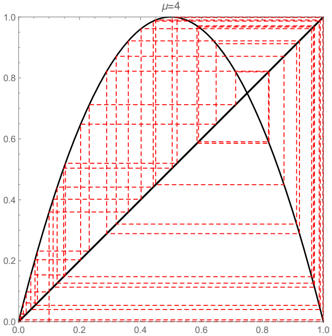

Figure: Graph of logistic map that exhibits chaos when $\mu > 3.56994$. In this case, we have $x_{n+1} = 4x_n(1 - x_n)$, where $\mu = 4$ and $x_0 = 0.1$.

Special Cases

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In the special cases where $\mu = 4$ and $\mu = 1$, we can find an *explicit solution* to calculate the n-th term of the recursion.

Special Cases

For $\mu = 4$

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$$x_n = \frac{1}{2}\left(1 - \cos(2^n \cos^{-1}(1 - 2x_0))\right)$$

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For $\mu = 1$

$$x_n = \frac{1}{2}\left(1 - e^{(2^n \ln(1-2x_0))}\right)$$

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Example of the $\mu = 4$ case:

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Say we want to find the third term of a recursion where $x_0 = 0.2$.

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Say we want to find the third term of a recursion where $x_0 = 0.2$.

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<u>or</u>

$x_3 = \frac{1}{2}(1 - \cos(2^3 \cos^{-1}(0.6))) \approx \frac{1}{2}(1 - \cos(425.041))$
$\approx \frac{1}{2}(0.578) = 0.289$

Conclusion

Difference equations offer a new perspective on the way we interpret recursive relations.

Through discrete dynamical systems, we can understand how expressions are influenced by equilibrium points and parameters of interest.

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