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The Duals of Warfield Groups

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THE DUALS OF WARFIELD GROUPS

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A Warfield group is a direct summand of a simply presented abelian group. In this paper, we describe the Pontrjagin dual groups of Warfield groups, both for the p-local and the general case. A variety of characterizations of these dual groups is obtained. In addition, numerical invariants are given that distinguish between two such groups which are not topologically isomorphic.

Introduction.

All considered groups in this paper will be abelian groups. By a p-local group we mean a module over $\mathbf{Z}_p$, the ring of integers localized at the prime p. Recall that an abelian group is said to be *simply presented* provided it can be defined in terms of generators and relations in such a way that all of the relations are of the form $mx = 0$ or $mx = y$. For instance, totally projective p-groups and completely decomposable torsion-free groups are simply presented. *Warfield groups* are direct summands of simply presented groups, in the p-local case they are also called *Warfield modules*. Warfield [**W**] established some characterizations of Warfield modules and gave a complete set of isomorphism invariants. Using an alternate definition, Hunter, Richman and Walker [**HRW1**], [**HRW2**], [**HRi**] proved existence theorems for p-local and global Warfield groups and were able to classify global Warfield groups. Moore [**M**] described Warfield modules in terms of quasi-sequentially nice submodules. Introducing the concept of a knice submodule, Hill and Megibben [**HM1**] characterized Warfield modules using the third axiom of countability. In [**HM3**], they proved a variety of characterizations of Warfield groups using Axiom 3 and the global definitions of nice and knice subgroups. Some more characterizations of Warfield groups are contained in [**L1**].

In [**F**], the following question was formulated as Problem 65: Which are the compact abelian groups whose duals are totally projective p-groups? Kiefer [**K**] described those groups dualizing the various characterizations of totally projective p-groups. Since Warfield groups are generalizations of totally projective p-groups, it seems to be a natural question to ask for the structure of the duals of Warfield groups. The aim of this paper is, both for the p-local and the general case, to dualize the concepts which are used to characterize Warfield groups, and to construct isomorphism invariants for the dual groups. After providing the necessary tools for dualization, we obtain various characterizations of the duals of Warfield modules (Theorem 2.15). In Proposition 2.16 we classify those compact $\mathbf{Z}_p$-modules in terms of numerical invariants. The corresponding characterizations of the duals of (global) Warfield groups are obtained in a similar manner as in the p-local case (Theorem 3.10). Using the dual tensor product we establish a classification theorem for the duals of global Warfield groups (Theorem 3.12).

Throughout this paper, all groups will be locally compact abelian groups. $\hat{G}$ will denote the character group of the locally compact abelian group G, and the annihilator of the subset S of G in $\hat{G}$ will be denoted by $(\hat{G}, S)$. We let G_0 denote the identity component of G and tG the torsion part of G. If H is a closed subgroup of G and n is a positive integer, then we let $n^{-1}H = \{x \in G : nx \in H\}$. For the fundamental results concerning abelian groups and Pontrjagin duality, we may refer to the books [**F**] and [**HR**]. Mostly, we will follow the terminology used in [**F**], [**HM1**] and [**HM3**]. From now on, let G denote a discrete group and C a compact group.

1. Preliminaries.

Let p be a prime. For each ordinal α, a closed subgroup $C[p^\alpha]$ of C is defined:

$C[p^0] = 0$
$C[p^{\alpha+1}] = \{x \in C : px \in C[p^\alpha]\}$
$C[p^\alpha] = \overline{\sum_{\beta<\alpha} C[p^\beta]}$ if α is a limit ordinal.

We put $C[p^\infty] = \sum_\alpha C[p^\alpha]$. The *$p$-order* of an element $x \in C$, written $o_p(x)$, is defined to be α if $x \in C[p^\alpha] \setminus \sum_{\beta<\alpha} C[p^\beta]$ and to be ∞ if $x \notin C[p^\infty]$. We set $o_p(0) = 0$. Clearly continuous homomorphisms do not increase p-orders. Suppose that a nonzero element x of C can be written as $x = p^n y = p^n z$. Since $o_p(y) = o_p(z)$, we may define $o_p(p^{-n}x)$ to be $o_p(y)$. If $\{H_i\}_{i\in I}$ is a nonempty collection of subgroups of G, then we have

$$\left(\hat{G}, \bigcap_{i\in I} H_i\right) = \overline{\sum_{i\in I} (\hat{G}, H_i)},$$

and obtain therefore

$$\left(\hat{G}, p^\alpha G\right) = \hat{G}[p^\alpha]$$

for all ordinals α.

Observe that G is a $\mathbf{Z}_p$-module exactly if $\hat{G}$ is a $\mathbf{Z}_p$-module because multiplication by a prime q is an automorphism of G if and only if multiplication by q is an automorphism of $\hat{G}$. Therefore the duals of Warfield modules must be found within the class of compact $\mathbf{Z}_p$-modules. The dual group of $\mathbf{Z}_p$ is the **a**-adic solenoid Σ_a where **a** is the subsequence of $(2, 3, 4, \dots)$ such that each a_i is relatively prime to p. This solenoid is denoted by Σ_p. For any compact groups C and D, the *dual tensor product of C and D* is defined to be the group $C \otimes_d D$ consisting of all elements $(x_{\delta,\eta}) \in \prod_{(\delta,\eta)\in\hat{C}\times\hat{D}} \mathbf{R}/\mathbf{Z}$ such that

$$x_{\delta_1+\delta_2,\eta} = x_{\delta_1,\eta} + x_{\delta_2,\eta} \quad \text{and} \quad x_{\delta,\eta_1+\eta_2} = x_{\delta,\eta_1} + x_{\delta,\eta_2}$$

for all δ, δ_1, $\delta_2 \in \hat{C}$ and η, η_1, $\eta_2 \in \hat{D}$.

It follows that the dual of $C \otimes_d D$ is isomorphic to $\hat{C} \otimes \hat{D}$ (see [**L2**], Proposition 1.2). Hence the dual of the p-localization $G_p = G \otimes \mathbf{Z}_p$ may be identified with the group $\hat{G}_{(p)} = \hat{G} \otimes_d \Sigma_p$ which we call the *(p)-localization of $\hat{G}$*.

Suppose that G is a module over the ring R. Then we say that G *is of simple isomorphism type* provided every nonzero submodule of G is isomorphic to G. A smooth chain

$$0 = N_0 \subset \ldots \subset N_\alpha \subset \ldots \subset N_\lambda = G$$

of submodules is called a *composition series for G* if every quotient $N_{\alpha+1}/N_\alpha$ is of simple isomorphism type. In this paper, we use the term "composition series" for a p-local group ($R = \mathbf{Z}_p$) and for an arbitrary group ($R = \mathbf{Z}$).

2. Local Case.

Throughout this section, G will denote a discrete and C a compact module, and all modules are assumed to be $\mathbf{Z}_p$-modules for a fixed prime p. If S is a subset of a module, then $\langle S\rangle$ will denote the submodule generated by S. Recall that a *height sequence* is a sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$ where every α_i is an ordinal or the symbol ∞ and $\alpha_i < \alpha_{i+1}$ for all $i < \omega$, where it is understood that $\infty < \infty$. For any $k < \omega$, let $p^k\overline{\alpha} = \{\alpha_{i+k}\}_{i<\omega}$. The *height sequence of $x \in G$* is the sequence $\{|p^i x|_p\}_{i<\omega}$ where $|x|_p$ denotes the p-height of x computed in G. Let $G(\overline{\alpha}) = \{x \in G : |p^i x|_p \geq \alpha_i \text{ for all } i < \omega\}$. In case $\alpha_i \neq \infty$ for all $i < \omega$, $G(\overline{\alpha}^*)$ is defined to be $\langle x \in G(\overline{\alpha}) : |p^i x|_p > \alpha_i$ for infinitely many values of $i\rangle$. In case $\alpha_i = \infty$ for some $i < \omega$, let $G(\overline{\alpha}^*)$ be the torsion part of $G(\overline{\alpha})$. An element $x \in G$ is said to be *primitive* if $x \notin G(\overline{\alpha}^*)$ where $\overline{\alpha}$ is the height sequence of x. A direct sum of a family of independent submodules A_i of G is said to be a *valuated coproduct in G* if for each $x = \sum x_i$ $(x_i \in A_i)$ we have $|x|_p = \min\{|x_i|_p\}$. In other words, $(\bigoplus A_i) \cap G(\overline{\alpha}) = \bigoplus(A_i \cap G(\overline{\alpha}))$ for each height sequence $\overline{\alpha}$. A valuated coproduct $\bigoplus A_i$ in G is called a *$*$-valuated coproduct in G* if $(\bigoplus A_i) \cap G(\overline{\alpha}^*) = \bigoplus(A_i \cap G(\overline{\alpha}^*))$ for every height sequence $\overline{\alpha}$. A *decomposition basis for G*

is a set $X = \{x_i\}_{i\in I}$ of elements of G such that each x_i has infinite order, $\langle X\rangle = \bigoplus_{i\in I}\langle x_i\rangle$ is a valuated coproduct, and $G/\langle X\rangle$ is torsion.

Recall that a submodule N of G is a *nice submodule* provided $p^\alpha(G/N) = (p^\alpha G+N)/N$ for all ordinals α. The module G *satisfies Axiom* 3 *with respect to nice submodules* if G has a system $\mathfrak{C}$ of nice submodules such that

(i) $0 \in \mathfrak{C}$;

(ii) $\sum N_i \in \mathfrak{C}$ if each $N_i \in \mathfrak{C}$;

(iii) given any $N \in \mathfrak{C}$ and a countable subset S of G, there exists an $L \in \mathfrak{C}$ satisfying $\langle N, S\rangle \subset L$ and $|L/N| \le \aleph_0$.

A decomposition basis X for G is called *nice* if $\langle X\rangle$ is a nice submodule of G. Let N be a nice submodule of G. Then N is called *knice* if the following condition is satisfied: if S is a finite subset of G, then there is a finite collection of primitive elements $y_1, \dots, y_m$ and an $r < \omega$ such that $N \oplus \langle y_1\rangle \oplus \cdots \oplus \langle y_m\rangle$ is a $*$-valuated coproduct in G containing $p^r\langle S\rangle$. N is called *quasi-sequentially nice* if for every element $x \in G$ there is a $k < \omega$ such that the coset $p^k x + N$ contains an element having the same height sequence as $p^k x + N$. Note that knice submodules are quasi-sequentially nice. A short exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ of modules is said to be *sequentially pure* if the induced sequence $0 \to G_1(\overline{\alpha}) \to G_2(\overline{\alpha}) \to G_3(\overline{\alpha}) \to 0$ is exact for all height sequences $\overline{\alpha}$. G is called *sequentially-pure-projective* if G has the projective property relative to all sequentially pure sequences of modules.

Now we are ready to state the various characterizations of Warfield modules.

Theorem 2.1 ([**HM1**], [**M**] and [**W**]). *The following conditions are equivalent for a* $\mathbf{Z}_p$*-module* G*:*

(i) *G is a Warfield module;*

(ii) *G is sequentially-pure-projective;*

(iii) *G has a nice decomposition basis X such that $G/\langle X\rangle$ is simply presented;*

(iv) *G has a decomposition basis and satisfies Axiom* 3 *with respect to nice submodules;*

(v) *G satisfies Axiom* 3 *with respect to knice submodules;*

(vi) *G satisfies Griffith's version of Axiom* 3 *with respect to knice submodules;*

(vii) *G has a composition series consisting of quasi-sequentially nice submodules;*

(viii) *G has a composition series $\{N_\alpha\}_{\alpha<\lambda}$ of nice submodules such that if $N_{\alpha+1}/N_\alpha$ is infinite, then $N_{\alpha+1} = N_\alpha \oplus \langle x_\alpha\rangle$ is a valuated coproduct in G for some $x_\alpha \in G$;*

(ix) *G has a composition series of knice submodules satisfying the conditions in* (viii)*;*

(x) *G has a composition series consisting of knice submodules.*

Recall that the *Ulm invariants of* G are defined to be

$$f_G(\beta) = \dim_{\mathbf{Z}/p\mathbf{Z}} (p^\beta G)[p]/(p^{\beta+1}G)[p]$$

and

$$f_G(\infty) = \dim_{\mathbf{Z}/p\mathbf{Z}} (p^\infty G)[p]$$

where β are ordinals. We follow the construction of Warfield invariants as described in [**HRi**]. Let A and B be submodules of G where $A \subset B$. For each height sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$ we let $A[\overline{\alpha}] = \{x \in A : |p^i x|_p \geq \alpha_i \text{ for all } i < \omega\}$ and $A[\overline{\alpha}^*] = \langle x \in A[\overline{\alpha}] : |p^i x|_p \neq \alpha_i \text{ for infinitely many values of } i\rangle$ where the p-height $|x|_p$ of each $x \in A$ is computed in G. This gives rise to a sequence

$$\frac{A[\overline{\alpha}]}{A[\overline{\alpha}^*]} \longrightarrow \frac{A[p\overline{\alpha}]}{A[p\overline{\alpha}^*]} \longrightarrow \frac{A[p^2\overline{\alpha}]}{A[p^2\overline{\alpha}^*]} \longrightarrow \dots$$

whose direct limit is denoted by $W_A(\overline{\alpha})$. The $\overline{\alpha}$*-th Warfield invariant of* A is the cardinal number

$$w_A(\overline{\alpha}) = \dim W_A(\overline{\alpha})$$

where the dimension is over the field $\mathbf{Z}/p\mathbf{Z}$ (if $\alpha_i \neq \infty$ for all i) or $\mathbf{Q}$ (if $\alpha_i = \infty$ for some i). It is shown in [**S**] that

$$w_G(\overline{\alpha}) = \dim G(\overline{\alpha})/G(\overline{\alpha}^*).$$

The cokernel of the induced map $\phi : W_A(\overline{\alpha}) \to W_B(\overline{\alpha})$ is a vector space over $\mathbf{Z}/p\mathbf{Z}$ or $\mathbf{Q}$ whose dimension is called the $\overline{\alpha}$*-th Warfield invariant of* B *relative to* A.

Theorem 2.2 ([**W**]). *Two Warfield modules are isomorphic if and only if they have the same Ulm and Warfield invariants.*

For any height sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$, let $C_{\overline{\alpha}} = \overline{\sum_{i<\omega} p^i(C[p^{\alpha_i}])}$, that is, the closed subgroup of C generated by the set $\{x \in C : \text{there is an } i < \omega \text{ such that } x \text{ is } p^i\text{-divisible and } o_p(p^{-i}x) \leq \alpha_i\}$. If $\alpha_i \neq \infty$ for all $i < \omega$, then we define $C_{\overline{\alpha}^*}$ to be the set of all elements contained in $C_{\overline{\alpha}} + \overline{\sum_{i\in I} p^i(C[p^{\alpha_i+1}])}$ for every infinite subset I of ω, and otherwise we let $C_{\overline{\alpha}^*} = C_{\overline{\alpha}} + C_0$, where C_0 is the identity component of C.

Lemma 2.3. *Let* $\overline{\alpha}$ *be a height sequence. Then* $(\hat{G}, G(\overline{\alpha})) = \hat{G}_{\overline{\alpha}}$ *and* $(\hat{G}, G(\overline{\alpha}^*)) = \hat{G}_{\overline{\alpha}^*}$.

Proof. Let $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$. Since $G(\overline{\alpha})$ coincides with $\bigcap_{i<\omega} p^{-i}(p^{\alpha_i}G)$, we obtain

$$\left(\hat{G}, G(\overline{\alpha})\right) = \overline{\sum_{i<\omega} \left(\hat{G}, p^{-i}(p^{\alpha_i}G)\right)} = \overline{\sum_{i<\omega} p^i \left(\hat{G}[p^{\alpha_i}]\right)} = \hat{G}_{\overline{\alpha}}.$$

If $\alpha_i \neq \infty$ for all $i < \omega$, then $G(\overline{\alpha}^*)$ is the subgroup generated by all groups

$$G(\overline{\alpha}) \cap \bigcap_{i \in I} p^{-i}(p^{\alpha_i+1}G)$$

where I is an infinite subset of ω. Therefore $(\hat{G}, G(\overline{\alpha}^*))$ is the intersection of all subgroups of the form

$$\left(\hat{G}, G(\overline{\alpha})\right) + \overline{\sum_{i \in I} \left(\hat{G}, p^{-i}(p^{\alpha_i+1}G)\right)}$$

where I is an infinite subset of ω, hence we obtain

$$\left(\hat{G}, G(\overline{\alpha}^*)\right) = \bigcap \left[\hat{G}_{\overline{\alpha}} + \overline{\sum_{i \in I} p^i \left(\hat{G}[p^{\alpha_i+1}]\right)}\right] = \hat{G}_{\overline{\alpha}^*}.$$

If $\alpha_i = \infty$ for some $i < \omega$, then

$$\left(\hat{G}, G(\overline{\alpha}^*)\right) = \left(\hat{G}, G(\overline{\alpha}) \cap tG\right) = \left(\hat{G}, G(\overline{\alpha})\right) + \left(\hat{G}, tG\right) = \hat{G}_{\overline{\alpha}^*}.$$

□

Let $\overline{\alpha}$ be a height sequence and suppose that B is a submodule of C. Then we let $B_{[\overline{\alpha},C]} = B + C_{\overline{\alpha}}$. If $\alpha_i \neq \infty$ for all $i < \omega$, then we define $B_{[\overline{\alpha}^*,C]}$ to be the set of all elements contained in $B_{[\overline{\alpha},C]} + \overline{\sum_{i \in I} p^i(C[p^{\alpha_i+1}])}$ for every infinite subset I of ω, and otherwise we let $B_{[\overline{\alpha}^*,C]} = C$.

Lemma 2.4. *Let A be a submodule of G and $\overline{\alpha}$ a height sequence. Then $(\hat{G}, A[\overline{\alpha}]) = (\hat{G}, A)_{[\overline{\alpha},\hat{G}]}$ and $(\hat{G}, A[\overline{\alpha}^*]) = (\hat{G}, A)_{[\overline{\alpha}^*,\hat{G}]}$.*

Proof. Let $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$. Then we have

$$\left(\hat{G}, A[\overline{\alpha}]\right) = \left(\hat{G}, A\right) + \overline{\sum_{i<\omega} \left(\hat{G}, p^{-i}(p^{\alpha_i}G)\right)} = \left(\hat{G}, A\right)_{[\overline{\alpha},\hat{G}]}.$$

If $\alpha_i \neq \infty$ for all $i < \omega$, then $A[\overline{\alpha}^*]$ is the subgroup generated by all groups $A[\overline{\alpha}] \cap \bigcap_{i \in I} p^{-i}(p^{\alpha_i+1}G)$ where I is an infinite subset of ω, and it follows that

$$\left(\hat{G}, A[\overline{\alpha}^*]\right) = \bigcap \left[\left(\hat{G}, A[\overline{\alpha}]\right) + \overline{\sum_{i \in I} \left(\hat{G}, p^{-i}(p^{\alpha_i+1}G)\right)}\right] = \left(\hat{G}, A\right)_{[\overline{\alpha}^*,\hat{G}]}.$$

If $\alpha_i = \infty$ for some $i < \omega$, then $A[\overline{\alpha}^*] = 0$, hence $(\hat{G}, A[\overline{\alpha}^*]) = \hat{G} = (\hat{G}, A)_{[\overline{\alpha}^*, \hat{G}]}$. □

Kiefer [**K**] introduced the definition of a smart subgroup of a compact (p)-group which we extend to compact $\mathbf{Z}_p$-modules: A closed submodule F of C is *a smart submodule* if $F[p^\alpha] = F \cap C[p^\alpha]$ for all ordinals α.

Proposition 2.5. *A subgroup N of G is a nice submodule if and only if $(\hat{G}, N)$ is a smart submodule of $\hat{G}$.*

Proof. See [**K**]. □

Definition 2.6. A module C is called a *simply given Σ_p-group* if there are disjoint sets J and K with $I = J \cup K \neq \emptyset$, and a map $f : K \to I$ such that

$$C = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} \Sigma_p : px_i = 0 \ \text{ if } \ i \in J \ \text{ and } \ px_i = x_{f(i)} \ \text{ if } \ i \in K \right\}.$$

Proposition 2.7. *G is a simply presented module if and only if $\hat{G}$ is a simply given Σ_p-group.*

Proof. Note that in a simply presented module, a set X of generators can be chosen such that $px \in X$ whenever $x \in X$ and $px \neq 0$ (see [**W**], Lemma 2.1). Now use the same argument as in [**K**], p. 304. □

Definition 2.8. A module C is said to have a *quasi-decomposition* if there is a short exact sequence

$$0 \longrightarrow K \xrightarrow{\subseteq} C \xrightarrow{\varphi} \prod_{j \in I} G_j \longrightarrow 0$$

of compact groups such that

(i) K is 0-dimensional;

(ii) $G_j = \Sigma_p$ for all $j \in I$;

(iii) the induced exact sequences $0 \to K_j \xrightarrow{\subseteq} C \xrightarrow{\pi_j \varphi} G_j \to 0$ ($\pi_j : \prod G_i \to G_j$ is the j-th projection map) yield $K + C[p^\alpha] = \bigcap_{j \in I} (K_j + C[p^\alpha])$ for all ordinals α.

If in addition K is both a smart submodule of C and a simply given Σ_p-group, then we say that C *has a simply given quasi-decomposition.*

Proposition 2.9. *A module G has a decomposition basis if and only if $\hat{G}$ has a quasi-decomposition. Moreover, G has a nice decomposition basis X with simply presented quotient $G/\langle X \rangle$ if and only if $\hat{G}$ has a simply given quasi-decomposition.*

Proof. Suppose that G has a decomposition basis $X = \{x_j\}_{j\in I}$. Each diagram

$$\begin{array}{ccccc}
\langle X\rangle & \xrightarrow{\subset} & G & \longrightarrow & G/\langle X\rangle \\
{\scriptstyle\cup}\uparrow & & \| & & \uparrow \\
\langle x_j\rangle & \xrightarrow{\subset} & G & \longrightarrow & G/\langle x_j\rangle
\end{array}$$

yields a dual diagram

$$\begin{array}{ccccc}
\prod\langle x_i\rangle\hat{} & \xleftarrow{\varphi} & \hat{G} & \xleftarrow{\supset} & (\hat{G},\langle X\rangle) \\
{\scriptstyle\pi_j}\downarrow & & \| & & \downarrow{\scriptstyle\cap} \\
\langle x_j\rangle\hat{} & \longleftarrow & \hat{G} & \xleftarrow{\supset} & (\hat{G},\langle x_j\rangle)
\end{array}\quad .$$

The module $K = (\hat{G},\langle X\rangle)$ is 0-dimensional because $G/\langle X\rangle$ is torsion, and the dual of $\langle x_j\rangle$ is the solenoid Σ_p for every $j \in I$. Since $\bigoplus\langle x_j\rangle$ is a valuated coproduct in G, we obtain $K + \hat{G}[p^\alpha] = (\hat{G},(\sum\langle x_j\rangle)\cap p^\alpha G) = (\hat{G},\sum(\langle x_j\rangle \cap p^\alpha G)) = \bigcap((\hat{G},\langle x_j\rangle) + \hat{G}[p^\alpha])$ for each ordinal α, as desired. Moreover, if $\langle X\rangle$ is nice in G and $G/\langle X\rangle$ is simply presented, then K is smart and simply given by Propositions 2.5 and 2.7.

Conversely, suppose that we have an exact sequence $0 \to K \xrightarrow{\subset} \hat{G} \xrightarrow{\varphi} \prod G_j \to 0$ satisfying the corresponding conditions in Definition 2.8. Then we obtain diagrams

$$\begin{array}{ccccc}
K & \xrightarrow{\subset} & \hat{G} & \xrightarrow{\varphi} & \prod G_i \\
{\scriptstyle\cap}\downarrow & & \| & & \downarrow{\scriptstyle\pi_j} \\
K_j & \xrightarrow{\subset} & \hat{G} & \longrightarrow & G_j
\end{array}$$

which yield the dual diagrams

$$\begin{array}{ccccc}
\hat{K} & \longleftarrow & G & \xleftarrow{\varphi^*} & \bigoplus(G_i)\hat{} \\
\uparrow & & \| & & \uparrow{\scriptstyle\pi_j^*} \\
(K_j)\hat{} & \longleftarrow & G & \longleftarrow & (G_j)\hat{}
\end{array}\quad .$$

For every $j \in I$ we have $G_j = \Sigma_p$, so its dual group is $\mathbf{Z}_p$, and we write $\varphi^*((G_j)\hat{}) = \langle x_j\rangle$. Using similar arguments as before we conclude that $X = \{x_j\}_{j\in I}$ is a decomposition basis for G. Further, if K is both a smart submodule and a simply given Σ_p-group, then $\langle X\rangle = \varphi^*(\bigoplus(G_i)\hat{}) = (G, \ker\varphi) = (G, K)$ is nice in G by Proposition 2.5 and $G/\langle X\rangle \cong \hat{K}$ is simply given. This completes the proof. □

The next definition is the dual analogue to knice submodules.

Definition 2.10. A smart submodule F of C is called a *k-smart submodule of C* provided the following condition is satisfied: If C is an extension of a closed subgroup D by a torus $\mathbf{T}^n$ $(n < \omega)$, then there is an $r < \omega$ and an exact sequence

$$0 \longrightarrow K \xrightarrow{\subset} C \xrightarrow{\varphi} \prod_{j=1}^{m} G_j \longrightarrow 0$$

$(m \geq 1, G_1 = C/F, G_2 = \cdots = G_m = \Sigma_p)$ with $p^r K \subset D$ so that the induced sequences

$$0 \longrightarrow K_j \xrightarrow{\subset} C \xrightarrow{\pi_j \varphi} G_j \longrightarrow 0$$

$(\pi_j : \prod G_i \to G_j$ is the j-th projection map) yield

(i) $K_1 = F$;

(ii) for every height sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$ and every $j \geq 2$, $C_{\overline{\alpha}^*} \subset K_j$ implies $p^i(C[p^{\alpha_i+1}]) \subset K_j$ for infinitely many values of i;

(iii) for every height sequence $\overline{\alpha}$ we have $K + C_{\overline{\alpha}} = \bigcap_{j=1}^{m}(K_j + C_{\overline{\alpha}})$ and $K + C_{\overline{\alpha}^*} = \bigcap_{j=1}^{m}(K_j + C_{\overline{\alpha}^*})$.

Proposition 2.11. *A subgroup N of G is a knice submodule if and only if $(\hat{G}, N)$ is a k-smart submodule of $\hat{G}$.*

Proof. By Proposition 2.5, N is a nice submodule of G if and only if $(\hat{G}, N)$ is a smart submodule of $\hat{G}$. Now suppose that N is knice in G and let D be a closed subgroup of $\hat{G}$ such that $\hat{G}/D = \mathbf{T}^n$ for some $n < \omega$. Then D is the annihilator of some subgroup A of G. Since $\hat{A}$ is topologically isomorphic to $\hat{G}/(\hat{G}, A)$, we conclude that A is a finitely generated subgroup of G. By the definition of kniceness, there are primitive elements $y_2, \ldots, y_m \in G$ and an $r < \omega$ such that $N \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_m \rangle$ is a $*$-valuated coproduct in G containing $p^r A$. We let $N_1 = N$, $N_i = \langle y_i \rangle$ for $i = 2, \ldots, m$ and put $G_i = \hat{G}/(\hat{G}, N_i)$ for all i. Then G_i can be identified with the solenoid Σ_p for $i = 2, \ldots, m$. Each diagram

$$\begin{array}{ccccc} \bigoplus N_i & \xrightarrow{\subset} & G & \longrightarrow & G/\bigoplus N_i \\ \cup\uparrow & & \| & & \uparrow \\ N_j & \xrightarrow{\subset} & G & \longrightarrow & G/N_j \end{array}$$

induces the dual diagram

$$\begin{array}{ccccc} \prod G_i & \xleftarrow{\varphi} & \hat{G} & \xleftarrow{\supset} & (\hat{G}, \bigoplus N_i) \\ \pi_j\downarrow & & \| & & \downarrow\cap \\ G_j & \longleftarrow & \hat{G} & \xleftarrow{\supset} & (\hat{G}, N_j) \end{array} .$$

Since $p^r A$ is contained in $\bigoplus N_i$, the group $p^r(\hat{G}, \bigoplus N_i)$ is contained in $(\hat{G}, A)$, so it remains to show (ii) and (iii) in Definition 2.10. Note that an element $x \in G$ is primitive exactly if it has infinite order and for each height sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$, $x \in G(\overline{\alpha}^*)$ implies $|p^i x|_p > \alpha_i$ for infinitely many values of i (see [**HM1**], p. 715). If $\hat{G}_{\overline{\alpha}^*}$ is contained in $(\hat{G}, N_j)$ for some $j \geq 2$, then N_j is contained in $(G, \hat{G}_{\overline{\alpha}^*})$ which is equal to $G(\overline{\alpha}^*)$. It follows that

$$p^i\left(\hat{G}[p^{\alpha_i+1}]\right) = \left(\hat{G}, p^{-i}(p^{\alpha_i+1}G)\right) \subset \left(\hat{G}, N_j\right)$$

for infinitely many values of i, hence condition (ii) in 2.10 is satisfied. Since $\bigoplus N_j$ is a $*$-valuated coproduct we can use similar arguments as before and conclude that

$$K + \hat{G}_{\overline{\alpha}} = \bigcap\left[\left(\hat{G}, N_j\right) + \hat{G}_{\overline{\alpha}}\right] \quad \text{and} \quad K + \hat{G}_{\overline{\alpha}^*} = \bigcap\left[\left(\hat{G}, N_j\right) + \hat{G}_{\overline{\alpha}^*}\right].$$

Thus $(\hat{G}, N)$ is a k-smart submodule.

Conversely, suppose that $(\hat{G}, N)$ is a k-smart submodule of $\hat{G}$. We let S be a finite subset of G and write the subgroup generated by S as $Z \oplus H$ where $Z = \mathbf{Z}^n$ for some $n < \omega$ and H is a finite p-group. We let $D = (\hat{G}, Z)$ and may identify $\hat{G}/D$ with $\mathbf{T}^n$, so there is an $r < \omega$ and an exact sequence $0 \to K \xrightarrow{\subset} \hat{G} \xrightarrow{\varphi} \prod G_j \to 0$ as described in Definition 2.10. Hence we obtain diagrams

$$\begin{array}{ccccc} K & \xrightarrow{\subset} & \hat{G} & \xrightarrow{\varphi} & \prod G_i \\ \cap\downarrow & & \| & & \downarrow\pi_j \\ K_j & \xrightarrow{\subset} & \hat{G} & \longrightarrow & G_j \end{array}$$

and corresponding dual diagrams

$$\begin{array}{ccccc} \hat{K} & \longleftarrow & G & \xleftarrow{\varphi^*} & \bigoplus(G_i)\hat{} \\ \uparrow & & \| & & \uparrow\pi_j^* \\ (K_j)\hat{} & \longleftarrow & G & \longleftarrow & (G_j)\hat{} \end{array}.$$

We can write $\varphi^*(\bigoplus(G_i)\hat{}) = N_1 \oplus N_2 \oplus \ldots \oplus N_m \subset G$ where

$$N_1 = (\varphi^*\pi_1^*)((G_1)\hat{}) = (G, \ker(\pi_1\varphi)) = \left(G, \left(\hat{G}, N\right)\right) = N$$

and $N_j = (\varphi^*\pi_j^*)((G_j)\hat{}) \cong (G_j)\hat{} \cong \mathbf{Z}_p$ for all $j \geq 2$. Hence we may write $N_j = \langle y_j \rangle$ for all $j \geq 2$. To show that each element y_j is primitive, let $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$ be a height sequence and assume that $y_j \in G(\overline{\alpha}^*)$. Then we get

$$\hat{G}_{\overline{\alpha}^*} \subset \left(\hat{G}, \langle y_j \rangle\right) = \left(\hat{G}, \ker(\pi_j\varphi)\right) = K_j,$$

so by condition (ii) in 2.10 we have

$$\left(\hat{G}, p^{-i}(p^{\alpha_i+1}G)\right) = p^i \left(\hat{G}[p^{\alpha_i+1}]\right) \subset K_j = \left(\hat{G}, \langle y_j\rangle\right)$$

for infinitely many values of i. Hence, $p^i y_j \in p^{\alpha_i+1}G$ for infinitely many values of i, so by the remark in the first part of this proof, each element y_j is primitive. Furthermore, $p^r Z$ is contained in (G, K) which is equal to $N \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_m\rangle$, therefore $p^t\langle S\rangle$ is contained in $N \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_m\rangle$ for some $t < \omega$. Finally, in view of condition (iii) in 2.10 and Lemma 2.3 we conclude that $N \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_m\rangle$ is a $*$-valuated coproduct in G. Hence N is a knice submodule, as desired. □

The third axiom of countability yields dually a condition in terms of metrizability which motivated us to call it Axiom M.

Definition 2.12. Let C be a module. Then C *satisfies Axiom M with respect to smart* (*k-smart*) *submodules* if C has a system $\mathfrak{F}$ of smart (k-smart) submodules such that

(i) $C \in \mathfrak{F}$;

(ii) $\mathfrak{F}$ is closed under taking arbitrary intersections;

(iii) given any $F \in \mathfrak{F}$ and a closed submodule K of F such that F/K is metrizable, there exists a $D \in \mathfrak{F}$ contained in K such that F/D is metrizable.

Next we dualize the concept of a quasi-sequentially nice submodule. Note that a nice submodule N is quasi-sequentially nice in G if and only if for every infinite cyclic subgroup A of G there exists a $k < \omega$ such that

$$(G/N)(\overline{\alpha}) \cap (p^k A + N)/N = (G(\overline{\alpha}) + N)/N \cap (p^k A + N)/N$$

for every height sequence $\overline{\alpha}$.

Definition 2.13. Let F be a smart submodule of C. Then we call F $*$-*smart in* C if the following condition holds: Whenever C is an extension of a closed subgroup D by the circle group $\mathbf{T}$, there exists a $k < \omega$ such that

$$F_{\overline{\alpha}} + (p^{-k}D \cap F) = (C_{\overline{\alpha}} \cap F) + (p^{-k}D \cap F)$$

for all height sequences $\overline{\alpha}$.

Proposition 2.14. *Let N be a submodule of G. Then N is quasi-sequentially nice in G if and only if $(\hat{G}, N)$ is $*$-smart in $\hat{G}$.*

Proof. Again, N is nice exactly if $(\hat{G}, N)$ is smart. Now let $\overline{\alpha}$ be a height sequence and $\rho : (G/N)\hat{} \to (\hat{G}, N)$ the topological isomorphism induced by the natural map $G \to G/N$. Suppose that N is quasi-sequentially nice and assume that D is a closed subgroup of $\hat{G}$ such that $\hat{G}/D$ is topologically isomorphic to the circle group $\mathbf{T}$. Then we may write $D = (\hat{G}, A)$ for some subgroup A of G which is isomorphic to $\mathbf{Z}$. By our assumption, there is a $k < \omega$ such that $(G/N)(\overline{\alpha}) \cap (p^k A + N)/N = (G(\overline{\alpha}) + N)/N \cap (p^k A + N)/N$. Now ρ maps $((G/N)\hat{}, (G/N)(\overline{\alpha}) \cap (p^k A + N)/N)$ onto $(\hat{G}, N)_{\overline{\alpha}} + (\hat{G}, p^k A + N)$ which is equal to $(\hat{G}, N)_{\overline{\alpha}} + [p^{-k} D \cap (\hat{G}, N)]$. On the other hand, ρ maps $((G/N)\hat{}, (G(\overline{\alpha}) + N)/N \cap (p^k A + N)/N)$ onto $(\hat{G}, G(\overline{\alpha}) + N) + (\hat{G}, p^k A + N)$ which is the same as $[\hat{G}_{\overline{\alpha}} \cap (\hat{G}, N)] + [p^{-k} D \cap (\hat{G}, N)]$. Therefore $(\hat{G}, N)$ is $*$-smart in $\hat{G}$. The converse can be shown using similar arguments. □

We call a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of compact modules *sequentially cobalanced* if for every height sequence $\overline{\alpha}$ the induced sequence

$$0 \longrightarrow A/A_{\overline{\alpha}} \longrightarrow B/B_{\overline{\alpha}} \longrightarrow C/C_{\overline{\alpha}} \longrightarrow 0$$

is exact. A descending chain

$$C = F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots \supset F_\lambda = 0$$

of closed submodules of C is called a *descending composition series for C* if $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$ whenever α is a limit ordinal and if each quotient $F_\alpha/F_{\alpha+1}$ is either cyclic of order p or topologically isomorphic to Σ_p.

Now we are ready to formulate the dual analogue of the various characterizations of Warfield modules.

Theorem 2.15. *The following conditions are equivalent for a compact $\mathbf{Z}_p$-module C:*

(i) *C is the dual group of a Warfield module;*

(ii) *C is a direct factor of a simply given Σ_p-group;*

(iii) *C has the injective property relative to all sequentially cobalanced exact sequences of compact $\mathbf{Z}_p$-modules;*

(iv) *C has a simply given quasi-decomposition;*

(v) *C has a quasi-decomposition and satisfies Axiom* M *with respect to smart submodules;*

(vi) *C satisfies Axiom* M *with respect to k-smart submodules;*

(vii) *C has a descending composition series consisting of $*$-smart submodules;*

(viii) *C has a descending composition series $\{F_\alpha\}_{\alpha<\lambda}$ of smart submodules such that if $F_\alpha/F_{\alpha+1}$ is infinite, then there is a closed submodule D_α of C such that $C/F_{\alpha+1} = F_\alpha/F_{\alpha+1} \oplus D_\alpha/F_{\alpha+1}$ and $C/(F_{\alpha+1} + C[p^\beta]) = (F_\alpha + C[p^\beta])/(F_{\alpha+1} + C[p^\beta]) \oplus (D_\alpha + C[p^\beta])/(F_{\alpha+1} + C[p^\beta])$ for every ordinal β;*

(ix) *C has a descending composition series consisting of k-smart submodules satisfying the conditions in* (viii)*;*

(x) *C has a descending composition series consisting of k-smart submodules.*

Proof. We show that each statement is equivalent to (i) by dualizing the various characterizations given in Theorem 2.1.

(ii) is equivalent to (i) because of Proposition 2.7.

(iii). A short exact sequence of compact modules is sequentially cobalanced exactly if the induced dual sequence is sequentially pure.

(iv). By Proposition 2.9, C has a simply given quasi-decomposition if and only if $\hat{C}$ has a nice decomposition basis with simply presented cokernel.

(v) and (vi). Axiom M is the dualized version of Axiom 3, and the annihilators of smart (k-smart) submodules of C in $\hat{C}$ are exactly the nice (knice) submodules of $\hat{C}$ by Propositions 2.5 and 2.11.

To prove that the statements (vii)-(x) are equivalent to (i), note that $\{F_\alpha\}_{\alpha<\lambda}$ is a descending composition series for C exactly if $\{(\hat{C}, F_\alpha)\}_{\alpha<\lambda}$ is a composition series for $\hat{C}$. Observe that $F_\alpha/F_{\alpha+1}$ is infinite if and only if $(\hat{C}, F_{\alpha+1})/(\hat{C}, F_\alpha)$ is infinite. Now C contains a closed submodule D_α as in statement (viii) exactly if D_α is the annihilator of a cyclic submodule $\langle x_\alpha\rangle$ of $\hat{C}$ in C such that $(\hat{C}, F_{\alpha+1}) = (\hat{C}, F_\alpha) \oplus \langle x_\alpha\rangle$ is a valuated coproduct in $\hat{C}$. This completes the proof. □

Now let β be an ordinal and $\overline{\alpha}$ a height sequence. We define

$$\hat{f}_C(\beta) = \mathfrak{m} \quad \text{if} \;\; (pC + C[p^{\beta+1}])/(pC + C[p^\beta]) \cong \prod_{\mathfrak{m}} \mathbf{Z}/p\mathbf{Z}$$

and

$$\hat{f}_C(\infty) = \mathfrak{n} \quad \text{if} \;\; C/(pC + C[p^\infty]) \cong \prod_{\mathfrak{n}} \mathbf{Z}/p\mathbf{Z}.$$

Note that $\hat{f}_C(\beta) = f_{\hat{C}}(\beta)$ and $\hat{f}_C(\infty) = f_{\hat{C}}(\infty)$. It follows from Lemma 2.3 that the dual of $C_{\overline{\alpha}^*}/C_{\overline{\alpha}}$ is isomorphic to $\hat{C}(\overline{\alpha})/\hat{C}(\overline{\alpha}^*)$ which is a vector space over $\mathbf{Z}/p\mathbf{Z}$ or $\mathbf{Q}$. By duality, the quotient $C_{\overline{\alpha}^*}/C_{\overline{\alpha}}$ is either connected and can be identified with a product $\prod_{\mathfrak{k}} \mathbf{Z}/p\mathbf{Z}$, or totally disconnected and can be written as $\prod_{\mathfrak{k}} \hat{\mathbf{Q}}$. In either case we define $\hat{w}_C(\overline{\alpha})$ to be $\mathfrak{k}$. In view of Warfield's Classification Theorem 2.2 we obtain

Proposition 2.16. *Let C and D be direct factors of simply given Σ_p-groups. Then C is topologically isomorphic to D if and only if $\hat{f}_C(\infty) = \hat{f}_D(\infty)$, and for all ordinals β and height sequences $\overline{\alpha}$, $\hat{f}_C(\beta) = \hat{f}_D(\beta)$ and $\hat{w}_C(\overline{\alpha}) = \hat{w}_D(\overline{\alpha})$.*

3. Global Case.

From now on, G will be a discrete and C a compact group. For any subset S of a group, $\langle S\rangle$ will denote the sub*group* generated by S. Recall that the *height matrix* of an element $x \in G$ is the doubly infinite $\mathbf{P}\times\omega$ matrix $\|x\|$ having the p-height $|p^i x|_p$ as its (p,i) entry. Occasionally, we write $\|x\|_G$ to emphasize that the height matrix is computed in the group G. More generally, a *height matrix* is a matrix $M = [m_{p,i}]_{(p,i)\in\mathbf{P}\times\omega}$ where each entry is an ordinal or the symbol ∞ such that $m_{p,i} < m_{p,i+1}$ for all primes p and all $i < \omega$. Again, we put $\infty < \infty$. Let $\overline{\infty}$ denote the height matrix having ∞ for each of its entries. If n is a positive integer, let nM denote the height matrix having $m_{p,j+i}$ as its (p,i) entry where $j = |n|_p$, the p-height of n in $\mathbf{Z}$. We let M_p be the p-row $(m_{p,i})_{i<\omega}$ of the height matrix M, so $\|x\|_p$ is the height sequence of x at p. If $M = [m_{p,i}]$ and $N = [n_{p,i}]$, then we write $M \le N$ in case $m_{p,i} \le n_{p,i}$ for all (p,i). Two height matrices M and N are said to be *quasi-equivalent*, and we write $M \sim N$, provided there are positive integers m and n such that $mN \ge M$ and $nM \ge N$. For each height matrix M, let $G(M) = \{x \in G : \|x\| \ge M\}$. In case $M \not\sim \overline{\infty}$ we let $G(M^*) = \langle x \in G(M) : \|x\| \not\sim M\rangle$ and if $M \sim \overline{\infty}$, then $G(M^*)$ is defined to be the torsion part of $G(M)$. For each prime p and each height sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$ let $G(\overline{\alpha}^*, p) = \langle x \in G : |p^i x|_p \ge \alpha_i$ for all i and $|p^i x|_p \ne \alpha_i$ for infinitely many values of $i\rangle$. Further, let $G(M^*, p) = G(M) \cap [G(M^*) + G(M_p^*, p)]$. The global definition of a primitive element is the following: an element $x \in G$ is called *primitive* if, for each positive integer n, each height matrix $M = [m_{p,i}]$ and each prime p, $nx \in G(M^*, p)$ implies either $\|x\| \not\sim M$ or $|p^i nx|_p \ne m_{p,i}$ for infinitely many values of i. Recall that a direct sum of independent subgroups A_i of G is *a valuated coproduct in G* if for each $x = \sum x_i$ $(x_i \in A_i)$ we have $|x|_p = \min\{|x_i|_p\}$ for all primes p. This means that $(\bigoplus A_i) \cap G(M) = \bigoplus(A_i \cap G(M))$ for each height matrix M. A valuated coproduct $\bigoplus A_i$ is a $*$-*valuated coproduct in G* provided $(\bigoplus A_i) \cap F = \bigoplus(A_i \cap F)$ for all subgroups F of the form $G(M)$, $G(M^*)$,

$G(\overline{\alpha}^*, p)$, or $G(M^*, p)$. A subset $X = \{x_i\}_{i\in I}$ of G is called *a decomposition basis for* G if all elements x_i have infinite order, $\langle X\rangle = \bigoplus_{i\in I}\langle x_i\rangle$ is a valuated coproduct, and $G/\langle X\rangle$ is a torsion group.

The global definition of a nice subgroup was introduced in [**HM2**]: A subgroup N of G is *nice in* G if and only if

$$\frac{p^\alpha(G/N)}{(p^\alpha G + N)/N}[p] = 0$$

for all ordinals α and all primes p. A decomposition basis X for G is said to be *nice* if $\langle X\rangle$ is a nice subgroup of G. A nice subgroup N of G is called *knice* provided the following condition is satisfied: If S is a finite subset of G, then there are primitive elements $y_1, \ldots, y_m \in G$ and a positive integer n such that $N \oplus \langle y_1\rangle \oplus \ldots \oplus \langle y_m\rangle$ is a $*$-valuated coproduct that contains $n\langle S\rangle$. We call a nice subgroup N *quasi-sequentially nice* if for every $x \in G$ there is a positive integer n and a $z \in N$ such that $\|nx + z\|_G = \|nx + N\|_{G/N}$. Note that as in the p-local case, knice subgroups are quasi-sequentially nice. Recall that a short exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ of discrete groups is *sequentially pure* exactly if the induced sequence $0 \to G_1(M) \to G_2(M) \to G_3(M) \to 0$ is exact for all height matrices M. Then G is called *sequentially-pure-projective* if G has the projective property relative to all sequentially pure sequences of abelian groups.

Now we state the various characterizations of global Warfield groups:

Theorem 3.1 ([**HM3**], [**HRi**] and [**L1**]). *The following conditions are equivalent for a group G:*

(i) *G is a Warfield group;*

(ii) *G is sequentially-pure-projective;*

(iii) *G has a nice decomposition basis X such that $G/\langle X\rangle$ is simply presented;*

(iv) *G has a decomposition basis and satisfies Axiom* 3 *with respect to nice subgroups;*

(v) *G satisfies Axiom* 3 *with respect to knice subgroups;*

(vi) *G satisfies Griffith's version of Axiom* 3 *with respect to knice subgroups;*

(vii) *G has a composition series consisting of quasi-sequentially nice subgroups;*

(viii) *G has a composition series $\{N_\alpha\}_{\alpha<\lambda}$ of nice subgroups such that if $N_{\alpha+1}/N_\alpha$ is infinite, then $N_{\alpha+1} = N_\alpha \oplus \langle x_\alpha\rangle$ is a valuated coproduct in G for some $x_\alpha \in G$;*

(ix) *G has a composition series of knice subgroups satisfying the conditions in* (viii)*;*

(x) *G has a composition series consisting of knice subgroups.*

In [**HRi**] the global Warfield invariants are defined in terms of *types* which are the equivalence classes of mutually quasi-homomorphic cyclic valuated groups. All torsion cyclics have the same type τ_0. A partial order on the set of all types is induced and it follows that $\tau \leq \tau_0$ for all types τ. The type of an element $x \in G$ is defined to be the type of $\langle x \rangle$. This gives rise to the definition of the type subgroups

$$G(\tau) = \{x \in G : \text{ type } (x) \geq \tau\}$$

and

$$G(\tau^*) = \langle x \in G(\tau) : \text{ type } (x) > \tau \rangle.$$

For each type τ, prime p, and height sequence $\overline{\alpha}$, the (*global*) *Warfield invariant* $w_G(\tau, p, \overline{\alpha})$ of G is defined to be the $\overline{\alpha}$-th Warfield invariant of $G(\tau)_p$ relative to $G(\tau^*)_p$ (see last section). Note that $w_G(\tau_0, p, \overline{\alpha}) = 0$ (cf. [**HRi**], p. 558). For every height matrix M, let $G[M] = \{x \in G : n\|x\| \geq M$ for some positive integer $n\}$ and $G[M^*] = \langle x \in G[M] : \|x\| \not\sim M \rangle + tG$. Observe that $G[M] = G[N]$ and $G[M^*] = G[N^*]$ if $M \sim N$. It follows that

$$\{G(\tau) : \tau \text{ is a type, } \tau \neq \tau_0\} = \{G[M] : M \text{ is a height matrix}\}$$

and

$$\{G(\tau^*) : \tau \text{ is a type, } \tau \neq \tau_0\} = \{G[M^*] : M \text{ is a height matrix}\}.$$

In the last part of this section we will determine isomorphism invariants for the duals of Warfield groups. Instead of dealing with groups $G(\tau)$ and $G(\tau^*)$ it is more convenient to consider the groups $G[M]$ and $G[M^*]$. For each quasi-equivalence class $\mathfrak{M}$ of height matrices, prime p and height sequence $\overline{\alpha}$, let $w_G(\mathfrak{M}, p, \overline{\alpha})$ be the $\overline{\alpha}$-th Warfield invariant of $G[M]_p$ relative to $G[M^*]_p$, where $M \in \mathfrak{M}$. We formulate the classification theorem of global Warfield groups in the following form:

Theorem 3.2 ([**HRi**])**.** *Suppose that G and H are Warfield groups. Then G is isomorphic to H if and only if for all quasi-equivalence classes $\mathfrak{M}$ of height matrices, primes p, ordinals β, and height sequences $\overline{\alpha}$ we have $f_{G_p}(\infty) = f_{H_p}(\infty)$, $f_{G_p}(\beta) = f_{H_p}(\beta)$, and $w_G(\mathfrak{M}, p, \overline{\alpha}) = w_H(\mathfrak{M}, p, \overline{\alpha})$.*

Let $M = [m_{p,i}]_{(p,i) \in \mathbf{P} \times \omega}$ be a height matrix. Then we define the group C_M to be $\overline{\sum_{(p,i) \in \mathbf{P} \times \omega} p^i(C[p^{m_{p,i}}])}$. If $M \not\sim \overline{\infty}$, let C_{M^*} be the closed subgroup generated by C_M and $\langle x \in C$: there is a positive integer n such that x is

p^i-divisible and $o_p(p^{-i}x) \leq m_{p,i+|n|_p}+1$ for all $(p,i) \in \mathbf{P}\times\omega$ with $m_{p,i+|n|_p} \neq \infty\rangle$. In case $M \sim \overline{\infty}$, define $C_{M^*} = C_0 + C_M$. Now let p be a prime and $\overline{\alpha} = \{\alpha_i\}_{i<\omega}$ a height sequence. We let $C_{\overline{\alpha},p} = \overline{\sum_{i<\omega} p^i(C[p^{\alpha_i}])}$. If $\alpha_i \neq \infty$ for all i, then we let $C_{\overline{\alpha}^*,p}$ be the set of all elements contained in $C_{\overline{\alpha},p}+\overline{\sum_{i\in I} p^i(C[p^{\alpha_i+1}])}$ for every infinite subset I of ω, and otherwise we put $C_{\overline{\alpha}^*,p} = C$. Finally, let $C_{M^*,p} = C_M + (C_{M^*} \cap C_{M_p^*,p})$. Now we can describe the annihilators of the subgroups $G(M)$, $G(M^*)$, $G(\overline{\alpha}^*,p)$ and $G(M^*,p)$:

Lemma 3.3. *Let G be a group, p a prime, $\overline{\alpha}$ a height sequence and M a height matrix. Then we have $(\hat{G},G(M)) = \hat{G}_M$, $(\hat{G},G(M^*)) = \hat{G}_{M^*}$, $(\hat{G},G(\overline{\alpha}^*,p)) = \hat{G}_{\overline{\alpha}^*,p}$ and $(\hat{G},G(M^*,p)) = \hat{G}_{M^*,p}$.*

Proof. The proof is very similar to the proof of Lemma 2.3. □

In view of the global definition of a nice subgroup we call a closed subgroup F of C *a smart subgroup of C* if the factor group

$$\frac{F \cap C[p^\alpha]}{F[p^\alpha]}$$

is p-divisible for all ordinals α and all primes p.

Proposition 3.4. *A subgroup N of G is nice in G if and only if $(\hat{G},N)$ is smart in $\hat{G}$.*

Proof. Note that for any discrete group A, we have $A[p] = 0$ if and only if $p\hat{A} = \hat{A}$. Let α be an ordinal, p a prime and $\rho : (G/N)\hat{} \to (\hat{G},N)$ the canonical topological isomorphism. Then ρ maps $((G/N)\hat{},(p^\alpha G+N)/N)$ onto $(\hat{G},N)\cap\hat{G}[p^\alpha]$ and $((G/N)\hat{},p^\alpha(G/N))$ onto $(\hat{G},N)[p^\alpha]$. Therefore we obtain

$$\left(\frac{p^\alpha(G/N)}{(p^\alpha G+N)/N}\right)\hat{} \cong \frac{(\hat{G},N)\cap\hat{G}[p^\alpha]}{(\hat{G},N)[p^\alpha]}.$$

This completes the proof. □

Definition 3.5. A group C is called *simply given* if there are sets I, J_p, K_p, and maps $f_p : K \to I$ (p primes) satisfying

(i) I is nonempty;

(ii) for each prime p, J_p and K_p are disjoint subsets of I;

(iii) the sets J_p are pairwise disjoint;

(iv) $f_p(i) \neq f_q(i)$ whenever $i \in K_p \cap K_q$ and $p \neq q$;

such that C is the subgroup of $\prod_{i\in I}\mathbf{R}/\mathbf{Z}$ consisting of all elements $(x_i)_{i\in I}$ with

$$px_i = 0 \ \text{ if } \ i \in J_p \ \text{ and } \ px_i = x_{f_p(i)} \ \text{ if } \ i \in K_p$$

for all primes p.

Definition 3.6. We say that C *has a quasi-decomposition* if there is a short exact sequence

$$0 \longrightarrow K \stackrel{\subset}{\longrightarrow} C \stackrel{\varphi}{\longrightarrow} \prod_{j\in I} G_j \longrightarrow 0$$

of compact groups such that

(i) K is 0-dimensional;

(ii) $G_j = \mathbf{T}$ for all $j \in I$;

(iii) the induced exact sequences $0 \to K_j \stackrel{\subset}{\to} C \stackrel{\pi_j\circ\varphi}{\longrightarrow} G_j \to 0$ ($\pi_j : \prod G_i \to G_j$ is the j-th projection map) yield $K + C_M = \bigcap_{j\in I}(K_j + C_M)$ for all height matrices M.

If in addition K is both a smart subgroup of C and simply given, then C is said to have a *simply given quasi-decomposition.*

Now we state the global definition of a k-smart subgroup.

Definition 3.7. A smart subgroup F of C is called a *k-smart subgroup of C* provided the following condition is satisfied: If C is an extension of a closed subgroup D by a torus $\mathbf{T}^k$ ($k < \omega$), then there is a positive integer r and an exact sequence

$$0 \longrightarrow K \stackrel{\subset}{\longrightarrow} C \stackrel{\varphi}{\longrightarrow} \prod_{j=1}^{m} G_j \longrightarrow 0$$

($m \geq 1, G_1 = C/F, G_2 = \ldots = G_m = \mathbf{T}$) with $rK \subset D$ so that the induced sequences

$$0 \longrightarrow K_j \stackrel{\subset}{\longrightarrow} C \stackrel{\pi_j\circ\varphi}{\longrightarrow} G_j \longrightarrow 0$$

($\pi_j : \prod G_i \to G_j$ is the j-th projection map) yield

(i) $K_1 = F$;

(ii) for every positive integer n, height matrix $M = [m_{p,i}]_{(p,i)\in\mathbf{P}\times\omega}$, prime p, and $j \geq 2$ we have: If $nC_{M^*,p} \subset F_j$, then either

  (a) for every positive integer n' there is a prime q and an $i < \omega$ such that $m_{q,i+|n'|_q} \neq \infty$ and $q^i C[q^{m_{q,i+|n'|_q}+1}] \subset F_j$ or

  (b) $m_{p,i} \neq \infty$ for all $i < \omega$ and $np^i(C[p^{m_{p,i}+1}]) \subset F_j$ for infinitely many values of i;

(iii) we have $K + H = \bigcap_{j=1}^{m}(K_j + H)$ for all subgroups H of the form C_M, C_{M^*}, $C_{\overline{\alpha}^*,p}$ or $C_{M^*,p}$.

Proposition 3.8. *A subgroup N of G is knice in G if and only if $(\hat{G}, N)$ is k-smart in $\hat{G}$.*

Proof. The assertion follows from the proof of Proposition 2.11, combined with Lemma 3.3. □

It is clear how to formulate the global version of Axiom M: A group C *satisfies Axiom* M *with respect to smart* (*k-smart*) *subgroups* if C has a system $\mathfrak{F}$ of smart (k-smart) subgroups satisfying the corresponding conditions in Definition 2.12. A smart subgroup F of C is called $*$*-smart in* C if the following condition holds: Whenever C is an extension of a closed subgroup D by the circle group $\mathbf{T}$, there exists a positive integer n such that

$$F_M + (n^{-1}D \cap F) = (C_M \cap F) + (n^{-1}D \cap F)$$

for every height matrix M.

Proposition 3.9. *Let N be a subgroup of G. Then N is quasi-sequentially nice in G if and only if $(\hat{G}, N)$ is $*$-smart in $\hat{G}$.*

Proof. Note that N is quasi-sequentially nice in G if and only if N is nice and if for every infinite cyclic subgroup A of G there exists a positive integer n such that

$$(G/N)(M) \cap (nA + N)/N = (G(M) + N)/N \cap (nA + N)/N$$

for every height matrix M. Now use the same logic as in the proof of Proposition 2.14. □

We call a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of compact groups *sequentially cobalanced* if for every height matrix M the induced sequence

$$0 \longrightarrow A/A_M \longrightarrow B/B_M \longrightarrow C/C_M \longrightarrow 0$$

is exact. Finally, we need the global definition of a descending composition series: A descending chain

$$C = F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots \supset F_\lambda = 0$$

of closed subgroups of C is called a *descending composition series for C* if $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$ whenever α is a limit ordinal and if each quotient $F_\alpha/F_{\alpha+1}$ is either cyclic of prime order or topologically isomorphic to the circle group $\mathbf{T}$.

Now we have all the tools available to formulate the global version of Theorem 2.15.

Theorem 3.10. *The following conditions are equivalent for a compact group C:*

(i) *C is the dual group of a Warfield group;*

(ii) *C is a direct factor of a simply given group;*

(iii) *C has the injective property relative to all sequentially cobalanced exact sequences of compact groups;*

(iv) *C has a simply given quasi-decomposition;*

(v) *C has a quasi-decomposition and satisfies Axiom* M *with respect to smart subgroups;*

(vi) *C satisfies Axiom* M *with respect to k-smart subgroups;*

(vii) *C has a descending composition series consisting of $*$-smart subgroups;*

(viii) *C has a descending composition series $\{F_\alpha\}_{\alpha<\lambda}$ of smart subgroups such that if $F_\alpha/F_{\alpha+1}$ is infinite, then there is a closed subgroup D_α of C such that $C/F_{\alpha+1} = F_\alpha/F_{\alpha+1} \oplus D_\alpha/F_{\alpha+1}$ and $C/(F_{\alpha+1} + C[p^\beta]) = (F_\alpha + C[p^\beta])/(F_{\alpha+1} + C[p^\beta]) \oplus (D_\alpha + C[p^\beta])/(F_{\alpha+1} + C[p^\beta])$ for every prime p and every ordinal β;*

(ix) *C has a descending composition series consisting of k-smart subgroups satisfying the conditions in* (viii)*;*

(x) *C has a descending composition series consisting of k-smart subgroups.*

Proof. We use the dualized concepts of the characterizations given in Theorem 3.1 to show that each statement is equivalent to (i).

(ii) is equivalent to (i) because C is a direct factor of a simply given group if and only if $\hat{C}$ is a direct summand of a simply presented group.

(iii). Again, a short exact sequence is sequentially cobalanced exactly if the induced dual sequence is sequentially pure.

(iv). As in the p-local case we conclude that C has a simply given quasi-decomposition if and only if $\hat{C}$ has a nice decomposition basis X with simply presented quotient $\hat{G}/\langle X\rangle$.

(v) and (vi) follow from the duality between Axiom 3 and Axiom M, and Propositions 3.4 and 3.8.

(vii)-(x). A compact group has a descending composition series if and only if its dual has a composition series. Further, a closed subgroup of C is $*$-

smart exactly if its annihilator in $\hat{C}$ is quasi-sequentially nice by Proposition 3.9. The statement (viii) is equivalent to (i) because of the global definition of a valuated coproduct and the argument used in the proof of Theorem 2.15. □

For every height matrix $M = [m_{p,i}]_{(p,i)\in\mathbf{P}\times\omega}$ we let

$$C_{[M]} = \left(\overline{\sum_{(p,i)\in\mathbf{P}\times\omega} p^i(C[p^{m_{p,i}}])}\right)_0,$$

that is, the identity component of the closed subgroup generated by $\{x \in C :$ there is a pair $(p,i) \in \mathbf{P}\times\omega$ such that x is p^i-divisible and $o_p(p^{-i}x) \leq m_{p,i}\}$. We define $C_{[M^*]}$ to be the closed subgroup generated by $C_{[M]}$ and $\langle x \in C_0 :$ There is a positive integer n such that x is p^i-divisible and $o_p(p^{-i}x) \leq m_{p,i+|n|_p} + 1$ for all $(p,i) \in \mathbf{P}\times\omega$ with $m_{p,i+|n|_p} \neq \infty\rangle$.

Lemma 3.11. *Let M be a height matrix. Then $(\hat{G}, G[M]) = \hat{G}_{[M]}$ and $(\hat{G}, G[M^*]) = \hat{G}_{[M^*]}$.*

Proof. Let $M = [m_{p,i}]_{(p,i)\in\mathbf{P}\times\omega}$. To show the assertion, we use similar arguments as before. Since

$$G[M] = \sum_{0<n<\omega} n^{-1}\left(\bigcap_{(p,i)\in\mathbf{P}\times\omega} p^{-i}(p^{m_{p,i}}G)\right),$$

the group $(\hat{G}, G[M]) = \bigcap_n n(\hat{G}, \bigcap_{(p,i)} p^{-i}(p^{m_{p,i}}G))$ is equal to the identity component of the group $(\hat{G}, \bigcap_{(p,i)} p^{-i}(p^{m_{p,i}}G))$ (see [**HR**], Theorem 24.24). Hence we get

$$\left(\hat{G}, G[M]\right) = \left(\overline{\sum_{(p,i)} p^i\left(\hat{G}, p^{m_{p,i}}G\right)}\right)_0 = \hat{G}_{[M]},$$

as desired. Furthermore, we have

$$G[M^*] = \left(G[M] \cap \bigcap_{0<n<\omega}\left(\sum_{\substack{(p,i)\in\mathbf{P}\times\omega\\ m_{p,i+|n|_p}\neq\infty}} p^{-i}(p^{m_{p,i+|n|_p}+1}G)\right)\right) + tG$$

and therefore

$$\left(\hat{G}, G[M^*]\right) = \left(\hat{G}_{[M]} + \overline{\sum_n\left(\bigcap_{(p,i)} p^i\left(\hat{G}[p^{m_{p,i+|n|_p}+1}]\right)\right)}\right) \cap \hat{G}_0.$$

Using the modular law we get therefore $(\hat{G}, G[M^*]) = \hat{G}_{[M^*]}$, as claimed. □

Let p be a prime and $\overline{\alpha}$ a height sequence. Further, let $\mathfrak{M}$ be a quasi-equivalence class of height matrices and pick any $M \in \mathfrak{M}$. Recall that the (p)-localization of C is the $\mathbf{Z}_p$-module $C_{(p)} = C \otimes_d \Sigma_p$. By [**L2**], Proposition 2.3, (p)-localization preserves exactness of short exact sequences, so if N is a closed subgroup of C, then we may view $N_{(p)}$ as a subgroup of $C_{(p)}$. In order to determine the dual Warfield invariants we define for every $n < \omega$ the groups

$$X(C, M, p^n\overline{\alpha}) = \frac{(C_{[M]})_{(p)[p^n\overline{\alpha}^*, C_{(p)}]}}{(C_{[M]})_{(p)[p^n\overline{\alpha}, C_{(p)}]}}$$

and

$$X(C, M^*, p^n\overline{\alpha}) = \frac{(C_{[M^*]})_{(p)[p^n\overline{\alpha}^*, C_{(p)}]}}{(C_{[M^*]})_{(p)[p^n\overline{\alpha}, C_{(p)}]}}.$$

Then we obtain sequences

$$X(C, M, \overline{\alpha}) \longleftarrow X(C, M, p\overline{\alpha}) \longleftarrow X(C, M, p^2\overline{\alpha}) \longleftarrow \dots$$

and

$$X(C, M^*, \overline{\alpha}) \longleftarrow X(C, M^*, p\overline{\alpha}) \longleftarrow X(C, M^*, p^2\overline{\alpha}) \longleftarrow \dots$$

where the maps are the homomorphisms induced by multiplication with p. Hence we get inverse limits $Z_{C_{[M](p)}}(\overline{\alpha})$ and $Z_{C_{[M^*](p)}}(\overline{\alpha})$. We will show in the next proof that a canonical map

$$Z_{C_{[M](p)}}(\overline{\alpha}) \longrightarrow Z_{C_{[M^*](p)}}(\overline{\alpha})$$

is obtained whose kernel is denoted by K. Further, it will turn out that if $\alpha_i \neq \infty$ for all i, then we can write $K = \prod_{\mathfrak{k}} \mathbf{Z}/p\mathbf{Z}$ and otherwise we may identify K with a product of the form $\prod_{\mathfrak{k}} \hat{\mathbf{Q}}$. In either case we define $\hat{w}_C(\mathfrak{M}, p, \overline{\alpha})$ to be $\mathfrak{k}$.

Now we are ready to formulate the dual version of Theorem 3.2.

Theorem 3.12 *Let C and D be direct factors of simply given groups. Then C is topologically isomorphic to D if and only if for all quasi-equivalence classes $\mathfrak{M}$ of height matrices, primes p, ordinals β and height sequences $\overline{\alpha}$ we have $\hat{f}_{C_{(p)}}(\infty) = \hat{f}_{D_{(p)}}(\infty)$, $\hat{f}_{C_{(p)}}(\beta) = \hat{f}_{D_{(p)}}(\beta)$, and $\hat{w}_C(\mathfrak{M}, p, \overline{\alpha}) = \hat{w}_D(\mathfrak{M}, p, \overline{\alpha})$.*

Proof. We may regard C and D as dual groups of Warfield groups G and H. Let p be a prime and β an ordinal. Since the dual of G_p is $\hat{G}_{(p)}$, the dual of $(p^\beta(G_p))[p]/(p^{\beta+1}(G_p))[p]$ can be identified with

$$\left(p\left(\hat{G}_{(p)}\right)+\hat{G}_{(p)}[p^{\beta+1}]\right)\Big/\left(p\left(\hat{G}_{(p)}\right)+\hat{G}_{(p)}[p^{\beta}]\right),$$

hence $f_{G_p}(\beta)$ is equal to $\hat{f}_{\hat{G}_{(p)}}(\beta)$. Similarly, we have $f_{G_p}(\infty)=\hat{f}_{\hat{G}_{(p)}}(\infty)$. Now let $\mathfrak{M}$ be a quasi-equivalence class of height matrices, $M\in\mathfrak{M}$ and $\overline{\alpha}$ a height sequence. Let $A=G[M^*]_p$ and $B=G[M]_p$. Then the cokernel of the canonical map

$$\phi: W_A(\overline{\alpha})\longrightarrow W_B(\overline{\alpha})$$

is a vector space over $\mathbf{Z}/p\mathbf{Z}$ or $\mathbf{Q}$ (cf. Section 2) whose dimension is equal to the Warfield invariant $w_G(\mathfrak{M},p,\overline{\alpha})$. Using Lemmas 2.4 and 3.11 we obtain

$$\left(\frac{A[\overline{\alpha}]}{A[\overline{\alpha}^*]}\right)^{\hat{}}\cong\frac{((G_p)\hat{},A)_{[\overline{\alpha}^*,(G_p)\hat{}]}}{((G_p)\hat{},A)_{[\overline{\alpha},(G_p)\hat{}]}}\cong\frac{(\hat{G}_{[M^*]})_{(p)[\overline{\alpha}^*,\hat{G}_{(p)}]}}{(\hat{G}_{[M^*]})_{(p)[\overline{\alpha},\hat{G}_{(p)}]}}.$$

It follows that there are sequences

$$X\left(\hat{G},M^*,\overline{\alpha}\right)\longleftarrow X\left(\hat{G},M^*,p\overline{\alpha}\right)\longleftarrow X\left(\hat{G},M^*,p^2\overline{\alpha}\right)\longleftarrow\dots$$

and

$$X\left(\hat{G},M,\overline{\alpha}\right)\longleftarrow X\left(\hat{G},M,p\overline{\alpha}\right)\longleftarrow X\left(\hat{G},M,p^2\overline{\alpha}\right)\longleftarrow\dots.$$

We get inverse limits $Z_{\hat{G}_{[M^*](p)}}(\overline{\alpha})$ and $Z_{\hat{G}_{[M](p)}}(\overline{\alpha})$ which are the duals of $W_A(\overline{\alpha})$ and $W_B(\overline{\alpha})$. Furthermore, the map ϕ induces the adjoint map

$$\phi^*: Z_{\hat{G}_{[M](p)}}(\overline{\alpha})\longrightarrow Z_{\hat{G}_{[M^*](p)}}(\overline{\alpha})$$

whose kernel K is the dual of coker ϕ. Hence $\hat{w}_{\hat{G}}(\mathfrak{M},p,\overline{\alpha})$ coincides with the Warfield invariant $w_G(\mathfrak{M},p,\overline{\alpha})$. Since two Warfield groups are isomorphic exactly if they have the same Ulm and Warfield invariants, the proof is complete. □

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